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# Eisenstein cocycles and Stark units in case TR$_p$:

$k \geq 2$, $v = (v_1, v_2) \in \mathbb{Q}^2/\mathbb{Z}^2$, $z \in \mathfrak{h}$.

$$E_{k,v}(z) = \sum_{m,n \in \mathbb{Z}}{}' \frac{e(mv_1 + nv_2)}{(mz+n)^k} \qquad (m,n) \neq (0,0) \qquad e(x) = e^{2\pi i x}$$

$$E_{k,v}|_\gamma(z) = (cz+d)^{-k} E_{k,v}\left(\frac{az+b}{cz+d}\right) \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) = \Gamma$$

$$= E_{k,\gamma^{-1} v}(z)$$

In particular,

$E_{k,v}(z) \in M_k(\Gamma(N))$ where $N$ = common denominator of $(v_1, v_2)$.

$\mathcal{P}_k$ = homogeneous polys. of deg $k$ $\subseteq \mathbb{C}[x,y] =: \mathcal{P}$.

$\mathcal{P}_{\mathbb{Q}} = \mathbb{Q}[x,y]$.

$\gamma \in SL_2(\mathbb{Z})$, $(\gamma P)(x,y) = P((x\ y)\gamma)$.

<u>Siegel's formula:</u> $F$ real quad. field., $\mathfrak{f} \subset \mathcal{O}_F$,

$K = K_{\mathfrak{f}}$ ray class field of cond. $\mathfrak{f}$.

$R = \{\infty_1, \infty_2, \wp | \mathfrak{f}\}$. Fix $\mathfrak{a} \subseteq \mathcal{O}_F$, $(\mathfrak{a}, \mathfrak{f}) = 1$.

$\mathfrak{a}^{-1}\mathfrak{f} = \langle w_1, w_2 \rangle$, $w_1 \bar{w}_2 - \bar{w}_1 w_2 > 0$.

$P(x,y) = N\mathfrak{a} \cdot \mathrm{Norm}_{F/\mathbb{Q}}(xw_1 + yw_2) \in \mathcal{P}_{\mathbb{Q},2}$

$(w_1, w_2)\varepsilon = (w_1, w_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\langle \varepsilon \rangle = E(\mathfrak{f})$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$0 < \varepsilon < 1$.

Define $v \in \mathbb{Q}^2$ by $1 = v_1 w_1 + v_2 w_2$.

Thm (Siegel): Fix $\tau \in \mathfrak{h}$. For $r \geq 1$,

$$\zeta_{K/F,\mathfrak{R}}(\mathfrak{a}, 1-r) = \frac{(2r-1)!}{(2\pi i)^{2r}} \int_{\tau}^{\gamma\tau} P(z,1)\, E_{2r,v}(z)\,dz.$$

$$\mathcal{V} := \mathbb{Q}^2/\mathbb{Z}^2 - \{0\}$$

Define: $\Psi_\tau(\gamma) : \mathcal{P} \times \mathcal{V} \longrightarrow \mathbb{C}$ by for $P \in \mathcal{P}_d$, $v \in \mathcal{V}$

$$\Psi_\tau(\gamma)(P,v) = \frac{(d+1)!}{(2\pi i)^{d+2}} \int_{\tau}^{\gamma\tau} P(z,1)\, E_{d+2,v}(z)\,dz.$$

$M = \{ f : \mathcal{P} \times \mathcal{V} \longrightarrow \mathbb{C} :$ linear in $\mathcal{P}$ and satisfies a certain dist. relations in $\mathcal{V} \}$

$f \in M$, $\gamma \in \Gamma$, $(\gamma f)(P,v) = f(\gamma^t P, \gamma^{-1} v)$

Prop: $\Psi_\tau(AB) = \Psi_\tau(A) + A\Psi_\tau(B)$ $A, B \in \Gamma$.

i.e,

$$\Psi_\tau \in Z^1(\Gamma, M)$$

and furthermore, $[\Psi_\tau] \in H^1(\Gamma, M)$ does not depend on $\tau \in \mathfrak{h}$.

Smoothing Fix $\ell$ prime.

$$E^{(\ell)}_{k,v}(z) = \ell^{k-2}\left(E_{k,(\ell v_1, v_2)}(\ell z) - E_{k,v}(z)\right)$$

$$E^{(\ell)}_{k,v}\big|_\gamma = E^{(\ell)}_{k,\gamma^{-1}v} \quad \text{for } \gamma \in \Gamma_0(\ell),\ v \in \mathcal{V}_\ell$$

where $V_\ell = (\mathbb{Q}^2 - (\frac{1}{\ell}\mathbb{Z} \oplus \mathbb{Z}))/\mathbb{Z}^2$.

$\Psi_{\tau,\ell}(\gamma)(P,V) =$ as $\Psi_\tau$ with $E^{(\ell)}_{k,v}$ instead of $E_{k,v}$.

$\Psi_{\tau,\ell} \in Z^1(\Gamma_0(\ell), M_\ell)$ $\qquad M_\ell =$ same as $M$ w/ $V$ replaced by $V_\ell$.

Const. term of $E^{(\ell)}_{k,v}$ is $0$ at $\infty$ and at $\Gamma_0(\ell)\infty$.

$\Rightarrow$ we can take $\tau = \infty$ in our def., i.e., $\Psi_{\infty,\ell}$ makes sense. This is a "partial modular symbol"

Integrality Thm:

"Thm": (D.-Darmon) $\Psi_{\infty,\ell}(P,V) \in \mathbb{Z}[\frac{1}{\ell}]$ if $P \in \mathbb{Z}[\frac{1}{\ell}][x,y]$
and $P(v + \mathbb{Z}[\frac{1}{\ell}] \oplus \mathbb{Z}) \subseteq \mathbb{Z}[\frac{1}{\ell}]$.
$\Psi_{\infty,\ell}(1,v) \in \mathbb{Z}$ $\quad (\ell \geq 5)$

Siegel's Thm revisited:

Fix an ideal $\mathfrak{L} \subset \mathcal{O}_F$ s.t. $N\mathfrak{L} = \ell$ (assume exists)
$T = \{\mathfrak{L}\}$ $\qquad \mathfrak{a}^{-1}\mathfrak{f} = \langle w_1, w_2\rangle$, $\mathfrak{a}^{-1}\mathfrak{L}^{-1}\mathfrak{f} = \langle \frac{1}{\ell}w_1, w_2\rangle$

Cor: $\quad \zeta_{K/F,R,T}(\sigma_{\mathfrak{a}}, 1-r) = \Psi_{\infty,\ell}(\gamma)(P^{r-1}, v)$
$\in \mathbb{Z}[\frac{1}{\ell}]$

and is in $\mathbb{Z}$ if $r=1$ and $\ell \geq 5$.
(Coates-Sinnott)

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## Sczech's construction of an Eisenstein cocycle:

Bogus calculation:

$$\Psi_\tau(\gamma)(1,v) = \frac{1}{(2\pi i)^2} \int_\tau^{\gamma\tau} \sum_{m,n\in\mathbb{Z}}{}' \frac{e(mv_1+nv_2)}{(mz+n)^2}\, dz$$

$$\text{"}=\text{"} \frac{1}{(2\pi i)^2} \sum_{m,n\in\mathbb{Z}}{}' e(mv_1+nv_2) \underbrace{\int_\tau^{\gamma\tau} \frac{1}{(mz+n)^2}\, dz}_{= \frac{\gamma\tau-\tau}{(m(\gamma\tau)+n)(m\tau+n)}}$$

Formally plug in $\tau = r/s \in \mathbb{Q}$:

$$\sigma_1 = \binom{r}{s} \quad \sigma_2 = \gamma\binom{r}{s}, \quad \sigma = (\sigma_1, \sigma_2) \in M_2(\mathbb{Z})$$

$$\Phi(\gamma)(1,v) \text{ "}=\text{" } \frac{1}{(2\pi i)^2} \sum_{z=(m,n)\in\mathbb{Z}^2}{}' \frac{\det(\sigma)}{\langle z,\sigma_1\rangle\langle z,\sigma_2\rangle} e(\langle z,v\rangle)$$

Problems:
- denom $= 0$?
- convergence?

These problems still need to be addressed. See online notes.