Algebraic cycles and Stark-Heegner points

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February 7, 2011

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Introduction

Stark-Heegner points are canonical global points on elliptic curves or modular abelian varieties which admit–often conjecturally–*explicit analytic constructions* as well as direct relations to L-series, both complex and p-adic. The ultimate aim of these notes is to present a new approach to this theory based on algebraic cycles, Rankin triple product L-functions, and p-adic families of modular forms. It is hoped that the reader could eventally use this text to get an understanding of the authors' strategy and of how it relates to the influential point of view on Euler systems and p-adic L-functions pioneered by Kato and Perrin-Riou.

In their current, very preliminary incarnation, the notes are meant to serve as a companion to the authors' lectures at the 2011 Arizona Winter School, and their scope is more limited. In spite of this, they still include a broader range of topics than could adequately be covered in the lectures, where the authors strived to present a narrower selection of topics, hopefully in somewhat greater depth.

In particular, the AWS lectures only cover the material in Chapters 1 to 7, in which no p-adic methods are involved. The topics covered in the last chapters are only included because of the (circumstantial, indirect) support they provide for the authors' belief that the study of diagonal cycles on the product of three Kuga-Sato varieties could eventually shed new light on some of the more mysterious aspects of the Stark-Heegner point construction.

The survey concludes with a bibliography giving a complete[1] list of the articles about Stark-Heegner points that have appeared in the literature until now. These articles are assigned labels like [LRV] or [Gre09], corresponding to the authors' initials followed eventually by a year of publication for items having already appeared. To limit the length of this bibliography, each chapter also contains a shorter list of more local references that are germane but not directly related to Stark-Heegner points, numbered consecutively within the chapter in which they are cited.

[1] at the time of writing, and to the best of our incomplete and fallible understanding! The authors apologise in advance to anyone who feels their contributions were omitted, underplayed, or misrepresented, and welcome suggestions for clarifications or additions to the bibliography.

Chapter 1

The conjecture of Birch and Swinnerton-Dyer

1.1 Prelude: units in number fields

The problem of calculating the unit groups of real quadratic fields, which essentially amounts to solving Pell's equation

$$x^2 - Dy^2 = 1, \tag{1.1}$$

is among the most ancient Diophantine problems, with systematic approaches to it dating back to the 7th century Indian mathematician and astronomer Brahmagupta. The early algorithms for solving Pell's equation can all be reduced to the method of *continued fractions* in various guises (or, as Fermat saw it, to his general *method of descent*). Of great importance for this survey are the following two more sophisticated approaches which grew out of Dirichlet's seminal work on primes in arithmetic progressions:

1. Circular units. Let $\chi : (\mathbb{Z}/D\mathbb{Z})^\times \longrightarrow \pm 1$ be an even primitive Dirichlet character of conductor D, and let $F = \mathbb{Q}(\sqrt{D})$ be the real quadratic field attached to χ by class field theory. In [1.1], Dirichlet obtains the striking identity

$$\varepsilon_F^h = \prod_{m=1}^{D} (1 - e^{2\pi i m/D})^{\chi(m)}, \tag{1.2}$$

where ε_F is a fundamental unit of F of norm one, the integer $h \geq 1$ is the *class number* of F and $e^x = \sum_{n=0}^{\infty} x^n/n!$ is the usual exponential function. Implicit in this identity is the containment $F \subset \mathbb{Q}(e^{2\pi i/D})$, a special case of the Kronecker-Weber theorem that is central to Gauss's fourth proof of the law of quadratic reciprocity. The expressions in the right hand side of (1.2) are the simplest examples of *circular units*, which in general give rise to a systematic collection of units in abelian extensions of $\mathbb{Q}$ and are of fundamental importance in the classical theory of cyclotomic fields.

2. The class number formula. The analytic class number formula for the Dirichlet L-series $L(s, \chi)$ (combined with the functional equation relating its values at s and $1 - s$)

asserts that

$$2h \log \varepsilon_F = L'(0, \chi) = 2\sqrt{D}L(1, \chi). \tag{1.3}$$

One can therefore recover ε_F (or at least its $2h$-th power) by exponentiating the special value $L'(0, \chi)$.

Both the approaches based on circular units and on special values of L-series lead to poor algorithms for solving Pell's equation, the simpler continued fraction method being far superior in practical terms; as Dirichlet himself writes in [1.1],

> "Il est sans doute inutile d'ajouter que le mode de solution que nous allons indiquer, est beaucoup moins propre au calcul numérique que celui qui dérive de l'emploi des fractions continues et que cette nouvelle manière de résoudre l'équation $t^2 - pu^2 = 1$, ne doit être envisagée que sous le rapport théorique comme un rapprochement entre deux branches de la science des nombres."

The "two branches" of number theory which Dirichlet alludes to can be broadly construed as *arithmetic*–the theory of integers and discrete quantities– and *analysis*–concerned with limits and continuous quantities. These branches have become closely intertwined since the 19th Century. It is now abundantly clear that both (1.2) and (1.3) epitomize a vigorous theme in modern number theory: *the explicit construction, by analytic means, of solutions to naturally occurring Diophantine equations.* The goal of these notes is to give a (biased, and incomplete) survey of a few aspects of this classical theme, with special emphasis on the case where unit groups of real quadratic fields are replaced by Mordell-Weil groups of elliptic curves over various global fields.

Even though (1.2) and (1.3) are essentially equivalent (cf., exercise 1.1 below) it is nonetheless worthwhile to draw a clear distinction between the two. This is partly because the theory of circular units underlying (1.2) is notoriously difficult to extend to other settings. Only when the ground field $\mathbb{Q}$ is replaced by a quadratic imaginary field K are circular units known to admit satisfactory analogues: the so-called *elliptic units* obtained by evaluating the Dedekind eta-function at arguments in K. Extending (1.2) to abelian extensions of more general number fields is one of the most tantalising questions in number theory, with close ties to Hilbert's twelfth problem and explicit class field theory.

On the other hand, equation (1.3) is the simplest non-trivial instance of the analytic class number formula which admits a well-known formulation for general number fields. If K is such a number field with class number h_K, discriminant D_K and ring of integers $\mathcal{O}_K$, and r_1 and $2r_2$ denote the number of distinct real and complex embeddings of K respectively, then the unit group $\mathcal{O}_K^\times$ is a finitely generated group of the form

$$\mathcal{O}_K^\times \simeq (\mathbb{Z}/w_K\mathbb{Z}) \times \mathbb{Z}^r, \qquad \text{where } r = r_1 + r_2 - 1, \tag{1.4}$$

by Dirichlet's unit theorem. The integer w_K appearing in this equation is the number of roots of unity in K. The general class number formula asserts that

$$\operatorname{res}_{s=1} \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{|D_K|}}, \tag{1.5}$$

where $\zeta_K(s)$ is the Dedekind zeta-function of K, and R_K is the *regulator* of K obtained by choosing a system $\varepsilon_1, \ldots \varepsilon_r$ of generators for the units of K modulo torsion and taking the determinant of an $r \times r$ matrix of logarithms of these units relative to any set $v_1, \ldots, v_r$ of distinct embeddings of K into $\mathbb{R}$ or $\mathbb{C}$:

$$R_K := \det \left(\log |v_j(\varepsilon_i)| \right)_{1 \le i,j \le r}. \tag{1.6}$$

It is sometimes more suggestive to rephrase (1.5) in terms of the behaviour of $\zeta_K(s)$ at $s = 0$, by applying the functional equation relating its values at s and $1 - s$. The class number formula then becomes

$$\mathrm{ord}_{s=0}\, \zeta_K(s) = r, \qquad \text{and} \quad \lim_{s \to 0} s^{-r} \zeta_K(s) = \frac{h_K R_K}{\# \mathcal{O}_K^\times}. \tag{1.7}$$

When $r = 1$ and K has a real embedding, (1.7) leads to an analytic expression for a unit of K, essentially by exponentiating $\zeta_K'(0)$. These conditions on (r_1, r_2) impose severe restrictions, since they only hold for real quadratic and complex cubic fields. It was Stark who first realised that the construction of units from derivatives of L-series can be made significantly more general by exploiting the natural ("motivic") factorisation of $\zeta_K(s)$ into Artin L-series attached to irreducible representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. More precisely, let $\tilde{K}$ be the normal closure of K over $\mathbb{Q}$ with Galois group $G := \mathrm{Gal}(\tilde{K}/\mathbb{Q})$, set $H := \mathrm{Gal}(\tilde{K}/K) \subset G$, and let

$$\rho = \mathrm{Ind}_H^G(\mathbf{1}) = \sum_i m_i \rho_i \tag{1.8}$$

be the induced representation, written as a direct sum of distinct irreducible representations ρ_i with multiplicities $m_i \ge 0$. Then $\zeta_K(s)$ is equal to the Artin L-series $L(\rho, s)$, which factors according to the decomposition in (1.8),

$$\zeta_K(s) = L(\rho, s) = \prod_i L(\rho_i, s)^{m_i}. \tag{1.9}$$

Stark has conjectured that the leading terms of each of the factors $L(\rho_i, s)$ on the right can be written down explicitely in terms of a regulator involving the ρ_i-isotypic part of the unit group of $\tilde{K}$. In particular, when $L(\rho_i, 0) = 0$, the first derivative $L'(\rho_i, 0)$ should be expressed as an explicit combination of logarithms of units of $\tilde{K}$. In this way, one can hope to recover non-trivial units from the values of $L'(\rho_i, 0)$ when the latter are non-zero.

For example, if $\rho = \mathrm{Ind}_K^{\mathbb{Q}} \chi$ is induced from an abelian character $\chi : \mathrm{Gal}(H/K) \longrightarrow \mathbb{C}^\times$, then

$$\mathrm{ord}_{s=0}\, L(\rho, s) = \mathrm{ord}_{s=0}\, L(H/K, \chi, s) = \#\{\text{Archimedean places } v \text{ of } K \text{ with } \chi(v) = 1\},$$

and one obtains non-trivial analytic constructions of units in H whenever this explicit quantity is equal to 1. This is the basis for Stark's striking conjectures which allow the construction of abelian extensions of number fields by analytic means and shed some light on the problem of finding an explicit class field theory for number fields beyond the case of the rationals and imaginary quadratic fields.

We suggest the reader to work on the following exercises and to think about their possible analogues in the theory of elliptic curves we shall be developing in the remainder of these notes.

Exercise 1.1. Explain how equations (1.2) and (1.3) are essentially equivalent. (The explicit evaluation of abelian L-series at $s = 1$ can be found in many textbooks such as [1.2].)

Exercise 1.2. Use the functional equation of $\zeta_K(s)$ and (1.9) to show how (1.3) is implied by (1.5).

Exercise 1.3. * There are many analogies between the arithmetic of the ring of integers $\mathcal{O}_F$ of a real quadratic field F, and the ring $\mathcal{O}_K[1/p]$ where K is an imaginary quadratic field in which the rational prime p is split. (For example, the group of elements of norm 1 has rank one, for both types of rings.) To what extent can the discussion in this section be adapted when the ring $\mathcal{O}_F$ is replaced by $\mathcal{O}_K[1/p]$?

1.2 Rational points on elliptic curves

Let now F be an arbitrary number field with ring of integers $\mathcal{O}_F$. There are many analogies between the study of the unit group of F (in geometric language, the group of integral points on the multiplicative group scheme $\mathbf{G}_m$ over $\mathcal{O}_F$) carried in the previous section and the group of rational points on elliptic curves E/F (or equivalently, points on an integral model over $\mathcal{O}_F$; since elliptic curves are complete varieties there is no significant distinction between the two).

For example, just as in the Dirichlet unit theorem reviewed in the previous section, the Mordell-Weil theorem for elliptic curves asserts that the Mordell-Weil group $E(F)$ is *finitely generated*. (However, there is no simple formula for the rank r of this group analogous to (1.4): this is one source of the extra richness and allure of the theory of elliptic curves.) Let $\mathfrak{N} \subset \mathcal{O}_F$ denote the conductor of E.

In transposing the discussion of §1.1 to the setting where $\mathbf{G}_m/\mathcal{O}_F$ is replaced by E/F, the role of the Dedekind zeta function of F is now played by the Hasse-Weil L-series $L(E/F, s)$ which is defined by the Euler product

$$L(E/F, s) = \prod_{\wp \nmid \mathfrak{N}} (1 - a_\wp \mathrm{N}(\wp)^{-s} + \mathrm{N}(\wp)^{1-2s})^{-1} \prod_{\wp \mid \mathfrak{N}} (1 - a_\wp \mathrm{N}(\wp)^{-s}), \tag{1.10}$$

where $\wp$ runs over the set of non-zero prime ideals of $\mathcal{O}_F$ and $\mathrm{N}(\wp)$ stands for the cardinality of the residue field $\mathbb{F}_\wp := \mathcal{O}_F/\wp$. The parameters $a_\wp$ are integers which satisfy Hasse's inequality

$$|a_\wp| \leq 2\sqrt{N(\wp)}, \tag{1.11}$$

implying that $L(E/F, s)$ is absolutely convergent on the right half plane $\mathrm{Re}(s) > 3/2$.

They can be described in terms of the reduction of E at $\wp$ as

$$a_\wp = \begin{cases} \mathrm{N}(\wp) + 1 - |E(\mathbb{F}_\wp)| & \text{if } E \text{ has good reduction at } \wp, \text{ i.e., } \wp \nmid \mathfrak{N}, \\ \pm 1 & \text{if } E \text{ has multiplicative reduction at } \wp, \text{ i.e., } \mathrm{ord}_\wp(\mathfrak{N}) = 1, \\ 0 & \text{if } E \text{ has additive reduction at } \wp, \text{ i.e. } \mathrm{ord}_\wp(\mathfrak{N}) \geq 2. \end{cases}$$

More precisely, in the case $\mathrm{ord}_\wp(\mathfrak{N}) = 1$ we have $a_\wp = +1$ (resp. $a_\wp = -1$) if the multiplicative reduction is split (resp. non-split).

As is well understood,[1] the role of the class number formula is played by the *Birch and Swinnerton-Dyer conjecture* which asserts that

$$L(E/F, s) \text{ has an analytic continuation to all of } \mathbb{C} \tag{1.12}$$

and satisfies

$$\mathrm{ord}_{s=1} L(E/F, s) \stackrel{?}{=} r. \tag{1.13}$$

One of the striking consequences of this conjectured equality is the implication

$$L(E/F, 1) = 0 \stackrel{?}{\Longrightarrow} E(F) \text{ is infinite,} \tag{1.14}$$

which leads to the following challenge—which underlies many of the questions that are treated in these notes:

Question 1.4. *When $L(E/F, 1) = 0$, construct a point of infinite order on $E(F)$.*

More generally, if $\chi : \mathrm{Gal}(H/F) \longrightarrow \mathbb{C}^\times$ is an abelian character, one can consider the twisted L-series $L(E/F, \chi, s)$ and hope to find a point of infinite order in the χ-component of $E(H) \otimes \mathbb{C}$ when that L-series vanishes at $s = 1$. Producing such a point is very difficult in practice. Even worse, all the information that we presently have about question 1.4—much of it fragmentary and conjectural—only applies to the case where the relevant L-series admits motivic factors having simple zeros[2] at the central point.

In analogy with (1.7) the Birch and Swinnerton-Dyer conjecture also makes a prediction about the *leading term* of $L(E/F, s)$ at $s = 1$:

$$\lim_{s \to 1} (s-1)^{-r} L(E/F, s) \stackrel{?}{=} \frac{\#\text{Ш}(E/F) R_{E/F}}{\#E(F)^2_{\mathrm{tors}}} \prod_v c_v, \tag{1.15}$$

where

- $\text{Ш}(E/F)$ is the conjecturally finite Shafarevich-Tate group of E over F, which measures the difficulty of computing the Mordell-Weil group $E(F)$ by descent and plays the role of the class group in the analogy with (1.7).

[1] but see the remark in the bibliography at the end of this chapter

[2] The reader interested in the Birch and Swinnerton Dyer conjecture in higher (analytic) rank may at this stage wish to put down these notes and retreat to a quiet attic for prolonged reflection.

- The term $R_{E/F}$ is the *regulator* of E/F, and is defined by

$$R_{E/F} = \det\left(\langle P_i, P_j\rangle_{1\le i,j\le r}\right), \tag{1.16}$$

where $P_1, \dots P_r$ is a basis for $E(F)$ modulo torsion and $\langle\ ,\ \rangle : E(F) \times E(F) \longrightarrow \mathbb{R}$ is the *canonical Néron-Tate height*, which extends to a non-degenerate positive definite bilinear form on $E(F) \otimes \mathbb{R}$.

- The quantities c_v are expressions attached to each place v of K. They can be made to depend on the choice of an invariant differential $\omega \in \Omega^1(E/F)$, and for archimedean v are given by

$$c_v = \begin{cases} \int_{E(K_v)} \omega & \text{if } v \text{ is real;} \\ \int_{E(K_v)} \omega \wedge \bar{\omega} & \text{if } v \text{ is complex.} \end{cases}$$

When v is non-archimedean the quantities c_v are rational and equal to 1 for all but finitely many v. While the individual quantities c_v depend linearly on the choice of the global differential ω, the product of the c_v is independent of this choice by the product formula.

In particular, when $r = 1$, equation (1.15) becomes

$$L'(E/F, 1) \stackrel{?}{=} \frac{\prod_v c_v}{\# E(F)^2_{\mathrm{tors}}} \langle P_{E,F}, P_{E,F}\rangle, \tag{1.17}$$

where $P_{E,F}$ is a point of infinite order of $E(F)$, which is essentially a multiple of a Mordell-Weil generator of $E(F)$ by the square root of the putative order of the Shafarevich-Tate group. The application of the class number formula to constructing units that was described in§sec:pell does not carry over directly to the elliptic curve context, since it is not so clear how to recover a point from the knowledge of its Néron-Tate height. (See, however, the article [1.8].)

Yet the analogy with the construction of units via circular units or Stark's conjecture can still be developed in several ways.

Heegner points: In some (ostensibly very special) situations, the method based on (1.2) and the theory of circular units has a counterpart for elliptic curves. This is provided by the theory of *Heegner points* arising from modular parametrisations of elliptic curves by modular curves and Shimura curves. The theory of Heegner points and their relations to L-series is briefly summarised in Chapter 3.

p-adic L-functions. Secondly, even though one can not directly recover rational points on $E(F)$ from the leading terms of the associated Hasse-Weil L-series at the central point, this defect can in some measure be addressed by replacing the Hasse-Weil L-series by various p-adic avatars. In a number of situations where $r = 1$, the leading terms of these p-adic L-series are given by p-adic formal group logarithms of algebraic points on $E(F)$ instead of their heights. The first example of this kind of phenomenon was discovered by Karl Rubin in [Ru] for an elliptic curve E with complex multiplication by an imaginary quadratic field K. Rubin's formula expresses the (square of the) formal group logarithm of a rational point on E in terms of a special value of the Katz p-adic L-function attached to K. Chapter 9 surveys a few p-adic formulae of "Gross-Zagier type", including the formula obtained in [BDP1] which has a similar flavour to Rubin's result but applies to *arbitrary* elliptic curves over $\mathbb{Q}$. Its main virtue, from the perspective of these notes, is that of possessing a natural generalisation leading to a *conjectural* construction of rational points on elliptic curves defined over fields that cannot be tackled via the theory of complex multiplication. The resulting "Stark-Heegner point conjecture", which differs somewhat from other constructions that have appeared in the literature over the last ten years, is briefly and imprecisely surveyed in Chapter 10 of these notes.

References

The point of view taken in the first section of this chapter is adapted from the introduction to [Dar96]. This article also marks the first appearance of the term "Stark-Heegner points" in the literature, but it is *definitely not* recommended as an introduction to the concept: the first stab at a definition that it contains is overly recundite and largely superseded by later developments.

For a discussion of Dirichlet's approach to Pell's equation, the reader can do no better than consult the original reference:

1.1. G. Lejeune Dirichlet. *Sur la manière de résoudre l'équation $t^2 - pu^2 = 1$ au moyen des fonctions circulaires.* Journal für die reine und angewandte Mathematik. Volume 1837, Issue 17, Pages 286–290,

which also contains the quote by Dirichlet that is reproduced above. For an introduction to circular units and the role they play in the theory of cyclotomic fields, see

1.2. Washington, Lawrence C. Introduction to cyclotomic fields. Second edition. Graduate Texts in Mathematics, **83**. Springer-Verlag, New York, 1997.

The best reference for Stark's conjecture is probably Tate's monograph

1.3. Tate, J. Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$. Lecture notes edited by Dominique Bernardi and Norbert Schappacher. Progress in Mathematics, **47**. Birkhäuser Boston, Inc., Boston, MA, 1984. 143 pp.

For a relatively recent example where Stark's conjecture is used to compute units in certain abelian sextic extensions of complex cubic fields, along the lines described in this chapter, see

1.4. Dummit, D.S., Tangedal, B.A. and van Wamelen, P.B. *Stark's conjecture over complex cubic number fields.* Math. Comp. **73** (2004), no. 247, 1525–1546.

The Birch and Swinnerton Dyer conjecture is the main recurring theme of these notes, as it serves as the main source of motivation of most of the topics discussed in the remaining chapters. For this reason, most of the references to the major breakthroughs obtained in this exciting area of research during the last four decades are to be found in the local bibliographies of the following chapters and in the global bibliography at the end of the manuscript. For a nice introduction to the subject, we recommend reading

1.5. B. J. Birch, Elliptic curves over $\mathbb{Q}$: A progress report, 1969 Number Theory Institute (Proc. Sympos. Pure Math., Vol. XX, State Univ. New York, Stony Brook, N.Y., 1969), Amer. Math. Soc., Providence, R.I., 1971, pp. 396400.

and

1.6. W. A. Stein, *The Birch and Swinnerton-Dyer Conjecture, a Computational Approach*. Dowloadable at
`http://modular.math.washington.edu/papers`.

There are many ways in which the analogy between the class number formula and the Birch and Swinnerton Dyer conjecture is not a perfect one. For example, the rank of the Mordell-Weil group is encoded in the *center of symmetry* $s = 1$ of its L-series, whereas in the setting of integral points on $\mathbb{G}_m$, the rank of the corresponding group is encoded in the two critical values $s = 0, 1$ which are interchanged by the functional equation and do not lie in the center of symmetry $s = 1/2$. For a nice introduction to the general framework (developped by Deligne and Bloch-Beilinson) underlying a whole array of conjectures relating special values of L-functions to arithmetic invariants, see

1.7. Minhyong Kim, *An introduction to motives I: classical motives and motivic L-functions*, Lectures at IHES summer school on motives, 2006. Dowloadable at
`http://www.ucl.ac.uk/ ucahmki/ihes3.pdf`.

For a discussion of how an *a priori* knowledge of the Néron-Tate height of a rational point on an elliptic curve can be used to assist in calculating it efficiently, see

1.8. Silverman, J.H. *Computing rational points on rank* 1 *elliptic curves via L-series and canonical heights*. Math. Comp. **68** (1999), no. 226, 835–858.

Chapter 2

Modular forms and analytic continuation

Fix an elliptic curve E of conductor $\mathfrak{N}$ over a number field F, as in the previous chapter. As recalled in (1.12) as part of the conjecture of Birch and Swinnerton-Dyer, $L(E/F,s)$ is predicted to extend to an entire function. Without this, (1.13) and (1.15) do not even make sense. The analytic continuation of $L(E/F,s)$ is open for elliptic curves over general number fields, as at present the only systematic method for proving it rests on the existence of an automorphic representation $\pi = \pi_E$ of $\mathbf{GL}_2(\mathbb{A}_F)$ whose associated L-function $L(\pi, s)$, which again may be defined as an Euler product over the non-zero prime ideals of $\mathcal{O}_F$, satisfies

$$L(\pi, s - \frac{1}{2}) = L(E/F, s). \tag{2.1}$$

Indeed, the former is known to extend to an entire function. When (2.1) holds one says that E/F is *modular.*

Thanks to the work of Wiles [2.6], Taylor-Wiles [2.7], Breuil-Conrad-Diamond-Taylor [2.8], Skinner-Wiles [2.9], and others, it is now known that all elliptic curves over $\mathbb{Q}$ and most elliptic curves over a totally real number field are modular. Their approach exploits the fact that, for such ground fields, the modularity of E and equality (2.1) can be rephrased in more geometric terms, as we now explain.

2.1 Quaternionic Shimura varieties

Assume F is totally real, of degree $n+1$ over $\mathbb{Q}$ for some $n \geq 0$. Let $\mathfrak{d}_F$ denote its different and $D_F = \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d}_F^{-1})$ its absolute discriminant. Let $\{v_i : F \hookrightarrow \mathbb{R}\}_{i=0}^n$ be the set of archimedean places of F, which we use to regard F as a subset of $\mathbb{R}^{n+1}$. Let also $\mathrm{Cl}^+(\mathcal{O}_F) = (F \otimes \hat{\mathbb{Z}})^\times / F^+ (\mathcal{O}_F \otimes \hat{\mathbb{Z}})^\times$ denote the narrow class group of F, where

$$F^+ = \{a \in F^\times, \quad v_i(a) > 0 \text{ for } i = 0, \dots, n\} \subset \mathbb{R}^{n+1}$$

is the subgroup of $F^\times$ of *totally positive elements.*

Let

$$B = F + Fi + Fj + Fij, \qquad ij = -ji, \quad i^2, j^2 \in F^\times$$

be a quaternion algebra, of reduced discriminant $\mathcal{D}_B \subseteq \mathcal{O}_F$. Letting $\mathrm{n} : B \to F$ denote the reduced norm, we write as before

$$B^+ := \{b \in B^\times, \mathrm{n}(b) \in F^+\} \subset B^\times.$$

Fix an ideal $\mathfrak{M} \subseteq \mathcal{O}_F$ which is prime to $\mathcal{D}_B$, and choose an Eichler order $R_0(\mathfrak{M})$ of level $\mathfrak{M}$ in B, that is, an order which is locally maximal at primes $\wp \mid \mathcal{D}_B$ and such that

$$R_0(\mathfrak{M}) \otimes \mathcal{O}_{F_\wp} \simeq \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \mathfrak{M} \mid c\} \subseteq \mathbb{M}_2(\mathcal{O}_{F_\wp})$$

at primes $\wp \nmid \mathcal{D}_B$. Fix one such isomorphism for each prime $\wp \nmid \mathcal{D}_B$. Letting d denote the number of real places of F at which the quaternion algebra B is indefinite, we choose a labelling $\{v_0, \ldots, v_{d-1}, v_d, \ldots, v_n\}$ of this set of archimedean places in such a way that

$$B \otimes_{v_i : F \hookrightarrow \mathbb{R}} \mathbb{R} \simeq \begin{cases} \mathbb{M}_2(\mathbb{R}) & \text{for } 0 \leq i \leq d-1 \\ \mathbb{H} & \text{for } d \leq i \leq n, \end{cases}$$

and fix such an isomorphism for each $i = 0, \ldots, n$. Put

$$\mathrm{Ram}(B) = \mathrm{Ram}_{fin}(B) \cup \mathrm{Ram}_\infty(B) = \{\wp \mid \mathcal{D}_B\} \cup \{v_i, d \leq i \leq n\}.$$

Define $K_0^B(\mathfrak{M}) = \prod_v K_0^B(\mathfrak{M})_v$ to be the open compact subgroup of $(B \otimes \mathbb{A}_F)^\times$ given by

$$K_0^B(\mathfrak{M})_v = \begin{cases} (R_0(\mathfrak{M}) \otimes \mathcal{O}_{F_\wp})^\times & \text{at finite places } v \text{ associated with a prime } \wp \\ \mathrm{SO}_2(\mathbb{R}) & \text{if } v = v_i \text{ with } 0 \leq i \leq d-1; \\ \mathbb{H}^\times & \text{if } v = v_i \text{ with } d \leq i \leq n. \end{cases}$$

Define also

$$K_1^B(\mathfrak{M}) = \{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0^B(\mathfrak{M}), d - 1 \in \mathfrak{M}\}.$$

The Shimura varieties associated with these groups are defined as

$$Y_\epsilon^B(\mathfrak{M}) := B^\times \backslash (B \otimes \mathbb{A}_F)^\times / \mathbb{A}_F^\times K_\epsilon^B(\mathfrak{M}). \tag{2.2}$$

The most classical cases arise when $F = \mathbb{Q}$ and $B = \mathbb{M}_2(\mathbb{Q})$, where modular curves $X_0^{\mathbb{M}_2(\mathbb{Q})}(N\mathbb{Z})$ and $X_1^{\mathbb{M}_2(\mathbb{Q})}(N\mathbb{Z})$ are simply denoted $X_0(N)$ and $X_1(N)$. The double coset space (2.2), endowed with the topology induced by the product topology of each of the local factors, is naturally a complex analytic variety of dimension $d \geq 0$. More precisely, for $d = 0$, it is a disjoint set of points indexed by the classes of one-sided ideals of $R_0(\mathfrak{M})$.

As for $d > 0$, define

$$\Gamma_\epsilon^B(\mathfrak{M}, \mathfrak{a}) = \gamma_{\mathfrak{a}\delta_F}^{-1} K_\epsilon^B(\mathfrak{M}) \gamma_{\mathfrak{a}\delta_F} \cap B^+ \overset{v_0, \ldots, v_{d-1}}{\hookrightarrow} \prod_{i<d} \mathbf{GL}_2^+(\mathbb{R}), \quad \epsilon = 0, 1, \tag{2.3}$$

for any integral ideal $\mathfrak{a}$ of F, where $\gamma_{\mathfrak{a}\delta_F} \in (B \otimes \hat{\mathbb{Z}})^\times$ satisfies $\mathrm{n}(\gamma_{\mathfrak{a}\delta_F}) \cdot (\mathcal{O}_F \otimes \hat{\mathbb{Z}}) \cap F = \mathfrak{a}$.

Let $\mathcal{H} = \{\tau = x + yi \in \mathbb{C}, y > 0\}$ denote Poincaré's upper half-plane and choose a set $\{\mathfrak{a}_j\}_{j=1}^h$ of representative ideals of $\mathrm{Cl}^+(\mathcal{O}_F)$, with $\mathfrak{a}_1 = \mathcal{O}_F$. Then, there is an isomorphism of analytic varieties

$$Y_\epsilon^B(\mathfrak{M}) \xrightarrow{\sim} \bigsqcup_{j=1}^{h} Y_\epsilon^{B,j}(\mathfrak{M}), \quad Y_\epsilon^{B,j}(\mathfrak{M}) = \Gamma_\epsilon^B(\mathfrak{M}, \mathfrak{a}_j)\backslash\mathcal{H}^d, \tag{2.4}$$

induced by the map $[\gamma] = [\{\gamma_\wp\}_\wp, \{\gamma_{v_i}\}] \mapsto \tau = (\gamma_{v_0} \cdot i, \ldots, \gamma_{v_{d-1}} \cdot i)_j$ where the representative $\gamma \in (B \otimes \mathbb{A}_F)^\times$ has been chosen such that $\{\gamma_\wp\}_\wp = \gamma_{\mathfrak{a}_j\delta_F}$ for some j and $\gamma_\infty = (\gamma_{v_i})_{i<d} \in \prod_i \mathbf{GL}_2^+(\mathbb{R})$. The strong approximation theorem ensures that such a choice is possible.

The variety $Y_\epsilon^B(\mathfrak{M})$ is compact if and only if $\mathrm{Ram}(B) \neq \emptyset$. If $\mathrm{Ram}(B) = \emptyset$, then $B \simeq \mathbb{M}_2(F)$ and $Y_\epsilon(\mathfrak{M}) := Y_\epsilon^{\mathbb{M}_2(F)}(\mathfrak{M})$ can be compactified by adjoining the finite set $\Gamma_\epsilon(\mathfrak{M}, \mathfrak{a}_j)\backslash\mathbb{P}^1(F)$ of cusps to each $Y_\epsilon^j(\mathfrak{M})$.

In any case, the resulting compact manifold admits a canonical projective model $X_\epsilon^B(\mathfrak{M})$ of dimension d over the *reflex field* $F_B := \bar{\mathbb{Q}}^H$, where H is the open subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which leaves the set $\{v_i\}_{i<d}$ invariant. The reflex field F_B is equal to F if $d = 1$, and to $\mathbb{Q}$ if $d - n + 1$.

2.2 Modular forms on quaternion algebras

Of great importance for the question that concern us here are the spaces of global sections of certain sheaves of differential forms on the above Shimura varieties, which in view of (2.4) are given naturally by tuples of functions, one for each connected component. For any $k = (k_0, \ldots, k_{d-1}) \in \mathbb{Z}_{\geq 1}^d$, write $S_k^B(\mathfrak{M}) \subseteq M_k^B(\mathfrak{M})$ for the finite-dimensional complex vector space of automorphic (cusp)forms of weight k on $X_1^B(\mathfrak{M})$. These spaces are most accessible computationally when $d = 0$ or 1, thanks to the work of Dembelé, Donnelly, Greenberg and Voight (cf. [2.2] for details): when $d = 0$, the Shimura variety $X_1^B(\mathfrak{M})$ is simply a finite union of points and automorphic forms can be described by a finite amount of data; when $d = 1$, $X_1^B(\mathfrak{M})$ is a curve and there is an algorithm (which makes use of the knowledge of a fundamental domain of the curve and is based on a cohomological interpretation of classical modular symbols) that computes the space $S_k^B(\mathfrak{M})$. For $d > 0$, an automorphic (cusp)form in $M_k^B(\mathfrak{M})$ may be explicitly defined as a collection $f = \{f_j : \mathcal{H}^d \longrightarrow \mathbb{C}\}$ of holomorphic functions such that

1. $f_j(\gamma \cdot \tau) = \prod_i (c_i + d_i\tau_i)^{k_i} f(\tau)$ for all $\tau \in \mathcal{H}^d$ and all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\epsilon^B(\mathfrak{N}, \mathfrak{a}) \subset \mathbf{GL}_2(\mathbb{R})^d$;

2. the components f_j are holomorphic (resp. vanish) at the cusps –if there are any.

Note that condition 2 is empty when $B \not\simeq \mathbb{M}_2(F)$, so that $S_k^B(\mathfrak{M}) = M_k^B(\mathfrak{M})$.

The group $\Gamma_1^B(\mathfrak{M}, \mathfrak{a}_j)$ is a normal subgroup of $\Gamma_0^B(\mathfrak{M}, \mathfrak{a}_j)$ and the quotient, naturally isomorphic to $(\mathcal{O}_F/\mathfrak{M})^\times$, acts on $M_k^B(\mathfrak{M})$, leaving $S_k^B(\mathfrak{M})$ invariant. Write

$$\langle d\rangle : M_k(\mathfrak{M}) \to M_k(\mathfrak{M})$$

for the *diamond operator* associated this way to any $d \in (\mathcal{O}_F/\mathfrak{M})^\times$.

Let $\psi : (\mathcal{O}_F/\mathfrak{M})^\times \longrightarrow \mathbb{C}^\times$ be a Dirichlet character of conductor $\mathfrak{M}$. An automorphic form $f \in M_k(\mathfrak{M})$ is said to have Nebentypus ψ if

$$\langle d \rangle f = \psi(d) f, \qquad \text{for all } d \in (\mathcal{O}_F/\mathfrak{M})^\times.$$

One obtains natural decompositions

$$M_k(\mathfrak{M}) = \oplus_\psi M_k(\mathfrak{M}, \psi), \qquad S_k(\mathfrak{M}) = \oplus_\psi S_k(\mathfrak{M}, \psi),$$

where ψ runs over all Dirichlet characters of conductor $\mathfrak{M}$. When $\psi = 1$ one recovers the spaces of automorphic forms on $X_0^B(\mathfrak{M})$.

The space $S_k^B(\mathfrak{M})$ is endowed with the hermitian Petersson scalar product

$$(\,,\,)_k : S_k^B(\mathfrak{M}) \times S_k^B(\mathfrak{M}) \longrightarrow \mathbb{C},\ (f,g)_k := \mu(Y_1^B(\mathfrak{M})) \int_{Y_1^B(\mathfrak{N})} f(\tau)\overline{g(\tau)} \prod_{i<d} y_i^{k_i} d\mu(\tau)$$

with $\tau_i = x_i + y_i$ and $\mu(\tau) = \prod_i y_i^{-2} dx_i dy_i$. The Atkin-Lehner theory of newforms generalises to this context and yields an orthogonal decomposition

$$S_k(\mathfrak{M}) = S_k^{B,old}(\mathfrak{M}) \oplus S_k^{B,new}(\mathfrak{M}),$$

where $S_k^{B,old}(\mathfrak{M})$ is the subspace generated by all cuspforms of lower level $\mathfrak{M}_0 \mid \mathfrak{M}$ with $\mathfrak{M}_0 \neq \mathfrak{M}$, and $S_k^{B,new}(\mathfrak{M})$ is the orthogonal complement of $S_k^{B,old}(\mathfrak{M})$ with respect to the Petersson scalar product.

The space $M_k^B(\mathfrak{M})$ is also equipped with a linear action of an algebra

$$\mathbb{T} = \mathbb{Z}[\{T_\wp\}_{\wp\nmid\mathfrak{M}}, \{U_\wp, W_\wp\}_{\wp\mid\mathfrak{M}}] \tag{2.5}$$

of *Hecke operators* which leave each of $M_k(\mathfrak{M}, \psi)$, $S_k(\mathfrak{M})$ and $S_k^{new}(\mathfrak{M})$ invariant (cf. [2.2] or [2.5] for the explicit definition of these operators). An element $f \in S_k(\mathfrak{M})$ is called a *newform* if $f \in S_k^{new}(\mathfrak{M})$ and it is an eigenvector for all $T \in \mathbb{T}$.

Assume for simplicity that $k = (k, \ldots, k)$ is a parallel weight. One of the most fundamental and useful results in this theory is provided by the Jacquet-Langlands correspondence, which asserts that there is a Hecke-equivariant isomorphism

$$S_k^{B_1,new}(\mathfrak{M}_1) \simeq S_k^{B_2,new}(\mathfrak{M}_2) \tag{2.6}$$

for any two pairs $(B_1, \mathfrak{M}_1)$, $(B_2, \mathfrak{M}_2)$ such that $\mathcal{D}_1 \cdot \mathfrak{M}_1 = \mathcal{D}_2 \cdot \mathfrak{M}_2$.

The most classical case arises when $B \simeq \mathbb{M}_2(F)$. The space of Hilbert modular forms $M_k(\mathfrak{M}) := M_k^{\mathbb{M}_2(F)}(\mathfrak{M}) = \mathrm{Eis}_k(\mathfrak{N}) \oplus S_k(\mathfrak{M})$ decomposes as the orthogonal direct sum of the spaces of Einsestein series and Hilbert cuspforms, and a form $f \in M_k(\mathfrak{M})$ can be explicitly written down by means of its Fourier expansion

$$f = \left(f_j(\tau) = a_0(f_j) + \sum_{\nu \in \mathfrak{a}_j \cap F^+} a_\nu(f_j) e^{2\pi i \operatorname{Tr}_{F/\mathbb{Q}}(\nu \cdot \tau)} \right)_{j=1,\ldots,n},$$

with $a_0(f_j) = 0$ if f is cuspidal. The Fourier coefficients of f can be slightly modified in order to be regarded as functions on integral ideals $\mathfrak{a}$ of F: define

$$c_{\mathfrak{a}}(f) := a_\nu(f_j)\,\mathrm{N}(\mathfrak{a}_j)^{-k/2} \text{ and } c_{\mathfrak{a},0}(f) := a_0(f_j)\,\mathrm{N}(\mathfrak{a}_j)^{-k/2} \quad \text{where } \mathfrak{a} = \nu\mathfrak{a}_j^{-1}, \nu \in F^+. \quad (2.7)$$

A cuspform f is said to be normalised if $c_{\mathcal{O}_F}(f) = 1$. The main result of Atkin-Lehner's theory, as extended by Shimura, asserts that the set of normalised Hilbert newforms is a basis of $S_k^{new}(\mathfrak{N})$. In addition, if f and g are newforms whose associated $T_{\mathfrak{p}}$-eigenvalues are equal for all but finitely many primes $\wp \nmid \mathfrak{N}$, then $\mathbb{C}f = \mathbb{C}g$. Shimura also showed that conjugation of Fourier coefficients induces a well-defined action of $\mathrm{Aut}(\mathbb{C})$ on $M_k(\mathfrak{N})$, from which it follows that the field $K_f := \mathbb{Q}(\{c_{\mathfrak{a}}(f)\}_j)$ generated by the Fourier coefficients of a normalised newform is a number field of *finite* degree over $\mathbb{Q}$, which turns out to be totally real if $\psi = 1$, and a CM-field otherwise. Moreover, he also proved that $M_k(\mathfrak{N}) = M_k(\mathfrak{N}; A) \otimes \mathbb{C}$ for any subring A of $\mathbb{C}$, where

$$M_k(\mathfrak{N}; A) := \{f \in M_k(\mathfrak{N}), c_{\mathfrak{a},0}(f), c_{\mathfrak{a}}(f) \in A \text{ for all } \mathfrak{a}\}.$$

One can attach an L-series to a normalised newform $f \in S_k(\mathfrak{N}, \psi)$ and a Hecke character $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ of conductor N_χ as follows:

$$L(f,\chi,s) = \prod_{\wp\nmid\mathfrak{N}}(1-\chi(\wp)c_\wp(f)\,\mathrm{N}(\wp)^{-s}+\chi(\wp)^2\psi(\wp)\,\mathrm{N}(\wp)^{1-2s})^{-1}\prod_{\wp\mid\mathfrak{N}}(1-\chi(\wp)c_\wp(f)\,\mathrm{N}(\wp)^{-s})^{-1}, \quad (2.8)$$

where $\chi(\wp) := 0$ if $\wp \mid N_\chi$. By adjoining to the above product a suitable local Euler factor at infinity, this L-series can be completed to a holomorphic function $\Lambda(f,\chi,s)$ which admits an integral representation as a Mellin transform. This is used to show that $L(f,\chi,s)$ admits an analytic continuation to the whole of $\mathbb{C}$, with

$$\mathrm{ord}_{s=1} L(f,\chi,s) = \mathrm{ord}_{s=1} \Lambda(f,\chi,s),$$

and that $\Lambda(f,\chi,s)$ satisfies the functional equation

$$\Lambda(f,\chi,s) = \omega(f)\Lambda(\bar{f},\chi^{-1},k-s), \text{ for some } \omega(f,\chi) \in \mathbb{C}^\times, |\omega(f)| = 1. \quad (2.9)$$

If $\chi = 1$ we simply write $L(f,s) = L(f,\chi,s)$. If K_f is totally real and $\chi^2 = 1$, then $\Lambda(\bar{f},\chi^{-1},s) = \Lambda(f,\chi,s)$ and $\mathrm{sign}(f,\chi) := \omega(f,\chi) \in \{\pm 1\}$.

2.3 Modularity over totally real fields

The modularity of an elliptic curve E/F of conductor $\mathfrak{N}$ can be phrased as the existence of a normalised newform $f \in S_2^{new}(\mathfrak{N})$ of weight 2 and trivial Nebentypus such that

$$L(f,s) = L(E/F,s). \quad (2.10)$$

Hence, if E is modular, (1.12) holds: $L(E/F,s)$ extends to an entire function. Moreover, the completed L-series of E satisfies $\Lambda(E/F,s) = \mathrm{sign}(E/F)\Lambda(E/F,2-s)$, with

$$\mathrm{sign}(E/F) = (-1)^{n+1}w_{\mathfrak{N}} = (-1)^{r+n+1} \tag{2.11}$$

where $W_\wp(f) = w_\wp f$ with $w_\wp \in \{\pm 1\}$ for any $\wp \mid \mathfrak{N}$, $w_{\mathfrak{N}} := \prod_{\wp \mid \mathfrak{N}} w_\wp$ and

$$r = \sharp\{\wp \mid \mathfrak{N} \text{ at which } E \text{ has split multiplicative reduction}\}.$$

Modular elliptic curves E/F over totally real fields are, as we have seen, the basic class of elliptic curves for which one can show (1.12) and is set to face (1.13). This class of curves can be broadened slightly by considering the base change of E to a field extension K/F, say finite Galois, since in analogy with (1.8) we have

$$L(E/K,s) = L(E/F, r_{K/F}, s) = \prod_i L(E/F, \varrho_i, s)^{m_i}, \quad r_{K/F} = \sum_i \varrho_i^{m_i} \tag{2.12}$$

where $r_{K/F}$ is the regular representation of $\mathrm{Gal}(K/F)$. When $\mathrm{Gal}(K/F)$ is solvable, it can be shown by repeatedly applying Langlands' cyclic base change that $L(E/K,s)$ extends to an entire function and satisfies a functional equation as in (2.9).

As explained in §1, the conjecture of Birch and Swinnerton-Dyer predicts that

$$\mathrm{rank}\, E(K) \stackrel{?}{=} \mathrm{ord}_{s=1} L(E/K,s). \tag{2.13}$$

A well-known refined version of the conjecture suggests that, for each irreducible representation ϱ of $\mathrm{Gal}(K/F)$ as in (2.12),

$$\mathrm{ord}_{s=1} L(E/F,\varrho,s) \stackrel{?}{=} \langle \varrho, \varrho_E \rangle, \tag{2.14}$$

the multiplicity of ϱ in the natural representation $\varrho_E : \mathrm{Gal}(K/F) \longrightarrow \mathbf{GL}(E(K)\otimes\mathbb{C})$.

In the few cases that (2.13) has been verified systematically, it is always under the hypothesis that $\dim \varrho_i = 1$ and $\mathrm{ord}_{s=1} L(E/F,\varrho_i,s) \leq 1$ for all i in (2.12) and the proof proceeds by showing (2.14) for each ϱ_i. Yet more remarkably, when $\mathrm{ord}_{s=1} L(E/F,\psi,s) = 0$ for a character $\psi : \mathrm{Gal}(K/F) \to \mathbb{C}^\times$, most (but not all, one exception being Kato's approach to bounding the Selmer groups of elliptic curves and modular abelian varieties) of the proofs of the finiteness of

$$E^\psi(K) := \{P \in E(K)\otimes_{\mathbb{Q}} \mathbb{C}, P^\sigma = \psi(\sigma)P\}$$

resort to the consideration of an auxiliary abelian variety A over some other number field K' (or a collection of them) such that $\mathrm{ord}(A/K',s) = 1$. See [GZ86], [BD97], [BD01], [Lo] for several instances of this strategy.

This underscores one of the fundamental questions treated in this manuscript, on which we insist again:

Question 2.1. *If $L(E/K,1) = 0$, how can one construct a non-torsion point on $E(K)$?*

References

There are many excellent books covering the classical theory of elliptic modular forms. One of them, among the many we could recommend, is

2.1. F. Diamond, J. Shurman, *A first course in modular forms*, Grad. text Math. **228**, Springer Verlag, 2005.

As for Hilbert modular forms over totally real fields, in this chapter we have adopted the more traditional point of view which avoids the language of adèles; the reader may also consult the following texts for more complete treatises, covering the subject from different points of view.

2.2. L. Dembelé, J. Voight, *Explicit methods for Hilbert modular forms*, submitted.

2.3. M. Dimitrov, *Galois representations modulo p and cohomology of Hilbert modular varieties*, Ann. Scient. Éc. Norm. Sup. **38** (2005), 505–551.

2.4. C. P. Mok, *Exceptional Zero Conjecture for Hilbert modular forms*, Compositio Math. **145:1** (2009), 1–55.

2.5. G. Shimura, On the special values of the zeta functions associated with Hilbert modular forms, *Duke Math. J.* **45** (1978), 637-679.

The reader who is curious about the proof of modularity of elliptic curves may wish to consult the original articles

2.6. A. Wiles, *Modular elliptic curves and Fermat's last theorem.* Ann. of Math. (2) **141** (1995), no. 3, 443551.

2.7. R. Taylor, A. Wiles, *Ring-theoretic properties of certain Hecke algebras.* Ann. of Math. (2) **141** (1995), no. 3, 553572.

2.8. C. Breuil, B. Conrad, F. Diamond, R. Taylor, *On the modularity of elliptic curves over* $\mathbf{Q}$*: wild 3-adic exercises.* J. Amer. Math. Soc. 14 (2001), no. 4, 843939.

The literature on modularity of elliptic curves over totally real fields is in constant expansion. For a flavour of the results that have be obtained, see for example

2.9. C. M. Skinner, A. Wiles, *Nearly ordinary deformations of irreducible residual representations.* Ann. Fac. Sci. Toulouse Math. (6) 10 (2001), no. 1, 185215.

Although fully satisfactory proofs of (2.14) have been achieved so far only under the hypothesis that $\dim \varrho = 1$ and $\mathrm{ord}_{s=1} L(E/F, \varrho, s) \leq 1$, the reader may consult

2.10. V. Dokchitser, L-functions of non-abelian twists of elliptic curves, Cambridge Ph.D. thesis, 2005.

and

2.11. H. Darmon, Y. Tian, Heegner points over towers of Kummer extensions Canadian J. Math., **62**, No. 5 (2010) 1060-1082.

for progress towards (2.14) for irreducible representations of dimension ≥ 2. For explicit instances of (2.13) when the order of vanishing is greater than 1 we encourage the reader to read

2.12. W. A. Stein, Heegner Points on Rank Two Elliptic Curves, preprint 2010, available at http://modular.math.washington.edu/papers

Chapter 3

Heegner points

3.1 Definition and construction

Let E be a modular elliptic curve of conductor $\mathfrak{N}$ over a totally real field F of degree $n+1$ over $\mathbb{Q}$ and let K/F be a quadratic field extension such that $(\mathrm{Disc}(K/F), \mathfrak{N}) = 1$. Write

$$\mathfrak{N} = \mathfrak{N}^+ \cdot \mathfrak{N}^-, \tag{3.1}$$

with

- $\mathfrak{N}^+ = \prod \wp_i^{f_i}$, where either the prime ideal $\wp_i$ splits in K or f_i is even,
- $\mathfrak{N}^- = \prod \wp_i^{f_i}$, where the prime ideals $\wp_i$ remain inert in K and f_i are odd.

Let $\psi : \mathrm{Gal}(K^{ab}/K) \to \mathbb{C}^\times$ be a character of finite order and conductor relatively prime to $\mathfrak{N}$. Then the sign of $L(E/K, \psi, s)$ is

$$\mathrm{sign}(E/K) = \mathrm{sign}(E/K, \psi) = (-1)^{r_2(K) + \sharp\{\wp | \mathfrak{N}^-\}}, \tag{3.2}$$

independently of the choice of ψ. Note that this expression is even simpler than formula (2.11) for $\mathrm{sign}(E/F)$.

For any abelian extension H/K, (2.12) implies that $L(E/H, s) = \prod_\psi L(E/K, \psi, s)$, where the product runs over all characters ψ of $\mathrm{Gal}(H/K)$. Hence, if $\mathrm{sign}(E/K) = -1$ and H is unramified at the primes dividing $\mathfrak{N}$, (3.2) suggests that

$$\mathrm{rank}\, E(H) \overset{?}{\geq} [H : K]. \tag{3.3}$$

A source that one hopes to exploit in order to construct a supply of Heegner points accounting for (3.3) relies again on the modularity of E. Thanks to the work of Brylinski-Labesse (cf. e.g. §2.1 of [2.3]), there exists a newform $f \in S_2(\Gamma_0(\mathfrak{N})) = S_2(\mathfrak{N}, \psi = 1)$ of trivial Nebentypus such that for all prime numbers ℓ there is an isomorphism of $\mathrm{Gal}(\bar{\mathbb{Q}}/\tilde{F})$-modules

$$\otimes_{i=0}^n H^1_{et}({}^{\sigma_i}E \times \bar{\mathbb{Q}}, \mathbb{Q}_\ell)(1) \simeq H^{n+1}_{et}(X_0(\mathfrak{N}) \times \bar{\mathbb{Q}}, \mathbb{Q}_\ell)_f(n+1), \tag{3.4}$$

where $\tilde{F}$ is the Galois closure of F and $\mathrm{Hom}(F, \bar{\mathbb{Q}}) = \{\sigma_i\}_{i=0}^n$. Let us explain a bit better what do we mean with this assertion.

Here, for a smooth variety X over $\mathbb{Q}$, the groups[1]

$$H^*(X \times \bar{\mathbb{Q}}, \mathbb{C}) = H^*_{\mathrm{B}}(X \times \bar{\mathbb{Q}}, \mathbb{Z}) \otimes \mathbb{C}, \qquad H^*_{et}(X \times \bar{\mathbb{Q}}, \mathbb{Q}_\ell)(r)$$

stand, respectively, for the complex singular cohomology and the rth Tate twist of the ℓ-adic étale cohomology groups of X.

When $X = X_0(\mathfrak{N})$, these cohomology groups are equipped with a natural action of the Hecke algebra $\mathbb{T}$ introduced in (2.5) and there exists a monomorphism of $\mathbb{T}$-modules

$$S_k(\mathfrak{N}, \psi = 1) \hookrightarrow H^{n+1}(X_0(\mathfrak{N}) \times \bar{\mathbb{Q}}, \mathbb{C}). \tag{3.5}$$

Hence, the form f can naturally be regarded as an element in $H^{n+1}(X_0(\mathfrak{N}) \times \bar{\mathbb{Q}}, \mathbb{C})$ and, by the standard comparison theorems, in $H^{n+1}_{et}(X_0(\mathfrak{N}) \times \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$. For any $\mathbb{T}$-module M we denote by M_f the maximal quotient of M on which any $T \in \mathbb{T}$ acts as $T \cdot m = a_T(f) \cdot m$. This explains all the notation in (3.4).

Tate's conjectures predict that there exists a cycle $\Pi^? \in \mathrm{CH}^{n+1}(X_0(\mathfrak{N}) \times \prod_i {}^{\sigma_i}E)$ inducing the isomorphism (3.4). If true, one could hope to exploit this circumtance in order to transfer the natural supply of cycles existing on $X_0(\mathfrak{N})$ to points on E.

At present, Tate's conjecture is known for curves, thanks to the work of Faltings. One can thus apply this strategy for $n = 0$, i.e. for the classical modular curve $X_0(N)$, where $\mathfrak{N} = (N)$, with $N \geq 1$.

But notice that one can still manage to stay in the pleasant realm of curves when $n \geq 1$ *provided* there exists a quaternion algebra B over F such that

1. $\mathrm{Ram}_\infty(B) = \{v_i, i > 0\}$ so that $d = 1$, and
2. $\mathcal{D}_B \mid\mid \mathfrak{N}$, by what we mean that all prime factors of $\mathcal{D}_B$ divide *exactly* $\mathfrak{N}$.

Since quaternion algebras ramify at an even number of places and their reduced discriminant $\mathcal{D}_B$ is square-free, this happens exactly when the following so-called *Jacquet-Langlands hypothesis* is satisfied[2]:

$$(JL) \qquad \text{Either } n \text{ is } \textit{even}\text{, or there exists } \wp \mid\mid \mathfrak{N}.$$

Indeed, when (JL) holds, it follows from (2.6) and (3.4) that

$$H^1_{et}(E \times_F \bar{\mathbb{Q}}, \mathbb{Q}_\ell)(1) \simeq H^1_{et}(X_0^B(\mathfrak{N}/\mathcal{D}_B) \times_F \bar{\mathbb{Q}}, \mathbb{Q}_\ell)_f(1) \tag{3.6}$$

[1] These groups will be discussed at greater length in future versions of these notes; at present, Chapter 4 only reviews the basic background on the algebraic de Rham cohomology of a variety; we refer the reader who is unfamiliar with étale cohomology to [4.8].

[2] Hypothesis (JL) can be slightly weakened, by only requiring that either n is even or there exists a prime $\wp \mid \mathfrak{N}$ at which the local automorphic representation $\pi_{f,\wp}$ associated to f belongs to the discrete series. If $\wp$ divides $\mathfrak{N}$ exactly, then $\pi_{f,\wp}$ always belong to the discrete series. When n is odd (so that $[F : \mathbb{Q}] = n + 1$ is even), the prototypical class of elliptic curves which fail to satisfy (JL) are those with good reduction everywhere. See [Gel75] and [3.8] for more details.

as $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$-modules, and Faltings' theorem implies that there exists a surjective map

$$\pi : \mathrm{Jac}(X_0^B(\mathfrak{N}/\mathcal{D}_B)) \longrightarrow E \tag{3.7}$$

of abelian varieties defined over F.

This is known as a *modular parametrisation* of E, and one would hope for a construction of divisor classes $D \in \mathrm{Div}^0(X_0^B(\mathfrak{N}/\mathcal{D}_B))(H)$ such that $P = \pi(D) \in E(H)$ are the sought-after points discussed above.

It thus remains to discuss for *which* quadratic extensions K/F does this strategy work, and if it does, *which* quaternion algebra B must be considered in order to obtain a modular parametrization (3.7) giving rise to non-torsion points $P = \pi(D) \in E(H)$.

Let us explain how Shimura-Taniyama's theory of complex multiplication makes the above hope feasible when $r_2(K) = n + 1$, i.e., when K is a CM-field. Let K be a totally complex quadratic extension of F such that $(\mathrm{Disc}(K/F), \mathfrak{N}) = 1$ and factor $\mathfrak{N} = \mathfrak{N}^+ \cdot \mathfrak{N}^-$ as in (3.1).

Definition 3.1. The pair (E, K) is said to satisfy the *Heegner hypothesis* if the ideal $\mathfrak{N}^-$ is square-free and divisible by an even (resp. odd) number of primes when n is even (resp. odd).

The most well-known particular case of the Heegner hypothesis arises when

$$F = \mathbb{Q}, \quad (\mathrm{Disc}(K/\mathbb{Q}), N) = 1, \quad N = N^+, \quad N^- = 1. \tag{3.8}$$

Since this was the working hypothesis of the classical work of Gross-Zagier [GZ86], this particular instance of the Heegner hypothesis is commonly referred to as the *Gross-Zagier hypothesis.*

Assume from now on that the Heegner hypothesis holds.

Choose an ideal class $\mathfrak{a}_j \in \mathrm{Cl}^+(F)$ corresponding to a connected component of the curve $X_0^B(\mathfrak{N}/\mathcal{D}_B)$; put

$$R = \gamma_{\mathfrak{a}\delta_F}^{-1} R_0(\mathfrak{N}/\mathcal{D}_B)\gamma_{\mathfrak{a}\delta_F} \subset B \overset{v_0}{\hookrightarrow} \mathbb{M}_2(\mathbb{R})$$

and

$$\Gamma = \Gamma_0^B(\mathfrak{N}/\mathcal{D}_B, \mathfrak{a}_j).$$

For each $\tau \in \Gamma\backslash\mathcal{H}$, define

$$\mathcal{O}_\tau = \{\gamma \in R : \gamma \cdot \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \lambda_\tau \begin{pmatrix} \tau \\ 1 \end{pmatrix} \text{ for some } \lambda_\tau \in \mathbb{C}\}. \tag{3.9}$$

The rule $\tau \mapsto \lambda_\tau$ identifies $\mathcal{O}_\tau$ with a subring of $\mathbb{C}$, isomorphic to either $\mathcal{O}_F$ or an order in a totally complex quadratic extension K of F.

Letting $\mathcal{O}_c$ denote the order of conductor $c \subset \mathcal{O}_F$ in K, define

$$\mathrm{CM}_j(\mathcal{O}_c) := \{\tau \in \Gamma\backslash\mathcal{H} : \mathcal{O}_\tau \simeq \mathcal{O}_c\}, \quad \mathrm{CM}(\mathcal{O}_c) := \sqcup_{j=1}^h \mathrm{CM}_j(\mathcal{O}_c) \subset X_0^B(\mathfrak{N}/\mathcal{D}_B)(\mathbb{C}).$$

Theorem 3.2. *Assume* $(c, \mathfrak{N}) = 1$. *Then* *(i) (Eichler)* $\mathrm{CM}(\mathcal{O}_c) \neq \emptyset$ *if and only if* $\mathfrak{N}^- = \mathcal{D}_B$ *(and thus* $\mathfrak{N}/\mathcal{D}_B = \mathcal{N}^+$*).*

(ii) (Shimura) $\mathrm{Div}^0(\mathrm{CM}(\mathcal{O}_c)) \subset \mathrm{Jac}(X_0^B(\mathfrak{N}^+))(H_c)$*, and thus* $\pi\,\mathrm{Div}^0(\mathrm{CM}(\mathcal{O}_c)) \subset E(H_c)$*,*

where H_c/K *denotes the ring class field associated to* $\mathcal{O}_c$*.*

The above result of Eichler and Shimura provides a very precise answer to the inquiries posed above: in order to hope for an affirmative answer to Question 2.1 and, more precisely, for a supply of points on E accounting for (3.3) by the above means, one must necessarily choose K/F to satisfy the Heegner hypothesis and B such that $\mathcal{D}_B = \mathcal{N}^-$.

Fix such choices for the remainder of this section. Then the rational points of (ii) on E are called *Heegner points*; moreover, $\mathrm{ord}_{s=1} L(E/K,s) = \mathrm{ord}_{s=1} L(E/K,\psi,s)$ is odd for any character $\psi : \mathrm{Gal}(H_c/K) \longrightarrow \mathbb{C}^\times$.

Define $e_\psi := \sum_{\sigma\in\mathrm{Gal}(H_1/K)} \psi^{-1}(\sigma)\cdot\sigma \in \mathrm{End}(E(H_c)\otimes\mathbb{C})$; it is an *idempotent* of $E(H_c)\otimes\mathbb{C}$ whose image is $E^\psi(H_c)$.

Theorem 3.3 (Kolyvagin, Gross-Zagier, Zhang). *If* $\mathrm{ord}_{s=1} L(E/K,\psi,s) = 1$*,*

$$e_\psi(\pi\,\mathrm{Div}^0(\mathrm{CM}(\mathcal{O}_c)) \otimes \mathbb{C}) = E^\psi(H_c) = \mathbb{C}P_K^\psi$$

for some $P_K^\psi \in E(H_c) \otimes \mathbb{Q}$*. Otherwise,* $e_\psi(\pi\,\mathrm{Div}^0(\mathrm{CM}(\mathcal{O}_c)) \otimes \mathbb{C}) = 0$*.*

There are many different choices for P_K^ψ, all them being the same up to torsion points and multiplication by scalars. One choice is

$$P_K^\psi = \pi(D_{\psi,c}) \in E^\psi(H_c), \quad D_{\psi,c} = e_\psi([\tau_c] - WW_\infty[\tau_c]) \in \mathrm{Jac}(X_0^B(\mathfrak{N}^+))^\psi(H_c),$$

where τ_c is any point in $\mathrm{CM}(\mathcal{O}_c)$, W_∞ is complex conjugation on $X_0^B(\mathfrak{N}^+)(\mathbb{C})$ and W is any Atkin-Lehner involution

$$W \in \langle W_\wp, \wp \mid \mathfrak{N}\rangle \text{ of } X_0^B(\mathfrak{N}^+) \text{ such that } W(f) = \mathrm{sign}(E/F)f.$$

Another choice is the one that used Gross and Zagier, and Zhang, for proving their contributions to the above theorem. Gross and Zagier worked with $F = \mathbb{Q}$ and $B = \mathbb{M}_2(\mathbb{Q})$, and took

$$P_K^\psi = \pi(D_{\psi,c}), \quad D_{\psi,c} = e_\psi([\tau_c] - \infty) \tag{3.10}$$

where ∞ denotes the cusp at infinity on $X_0(N)$. In general, Zhang took $P_K^\psi = e_\psi\pi([\tau] - \xi)$ for a certain divisor ξ of degree 1 made out of cusps and elliptic points:

Theorem 3.4 (Gross-Zagier, Zhang). *Let* $\langle\,,\,\rangle$ *denote the canonical Néron-Tate height pairing. Then*

$$L'(E/K,\psi,1) \doteq (f,f)_2 \cdot \langle P_K^\psi, P_K^\psi\rangle.$$

Throughout the symbol $\doteq$ shall denote equality up to a non-zero fudge factor, which can be made explicit. This striking result of Gross, Zagier and Zhang proves half of Theorem 3.3 above: the point P_K^ψ is torsion if and only if $L'(E/K, \psi, 1) = 0$. Kolyvagin proved the other half: if P_K^ψ is not torsion, it generates the whole $E^\psi(H_c)$. Keeping the above notations and hypothesis, the following corollary is an immediate consequence of Theorem 3.3 and our discussion above.

Corollary 3.5. *Let H/K be a finite abelian extension, unramified at the primes dividing $\mathfrak{N}$, and Galois generalized dihedral over $\mathbb{Q}$. Then* $\mathrm{ord}_{s=1} L(E/H, s) \geq [H : K]$*; if* $\mathrm{ord}_{s=1} L(E/H, s) = [H : K]$ *then* $\mathrm{rank}\, E(H) = [H : K]$.

Indeed, a classical result of Bruckner shows that all such class fields H are contained in some ring class field H_c, $(c, \mathfrak{N}) = 1$. In addition one can also derive the following result.

Corollary 3.6. *Assume* $\mathrm{ord}_{s=1} L(E/F, s) \leq 1$. *Then* $\mathrm{rank}\, E(F) = \mathrm{ord}_{s=1} L(E/F, s)$.

This is slightly more subtle: the analytic non-vanishing results of Bump-Friedberg-Hoffstein produce a totally complex quadratic extension K/F satisfying condition (i) of Theorem 3.2, and for which $L(E/K, s)$ has a simple zero at $s = 1$. By Theorem 3.4, P_K generates $E(K)$, and its trace therefore generates $E(F)$. It is not hard to show that this trace vanishes when $L(E/F, 1) \neq 0$, and is of infinite order when $L(E/F, 1) = 0$.

3.2 Numerical calculation

In view of Theorem 3.3, one may wonder: *how can we explicitly compute $P_K^\psi \in E(H_c)$?*

When $\mathcal{D}_B \cdot \mathfrak{N}^+ = \mathfrak{N}$ is a bit large, one can not hope to dispose of an explicit, workable algebraic description of the curve $X_0^B(\mathfrak{N}^+)$, of the set of CM-points on them and of the modular parametrisation π. The only alternative that manifests itself is resorting to an *analytic description* of these three ingredients needed for the computation of P_K^ψ.

This is indeed available when $F = \mathbb{Q}$ and $B = \mathbb{M}_2(\mathbb{Q})$, since the classical modular parametrisation $X_0(N) \hookrightarrow \mathrm{Jac}(X_0(N)) \longrightarrow E$ sending the point P to $\pi(P - \infty)$ takes the shape

$$\tau \mapsto z_\tau := c \int_{i\infty}^{\tau} \omega_f = c \sum_{n \geq 1} \frac{a_n}{n} e^{2\pi i n \tau} \tag{3.11}$$

over $\mathbb{C}$, after letting $\tau \in \Gamma_0(N)\backslash\mathcal{H}$ be the element attached to the point P under the uniformisation $Y_0(N) = \Gamma_0(N)\backslash\mathcal{H}$. In (3.11), the expression ω_f denotes the regular differential $\pi i f(z)dz \in H^0(\Omega^1_{X_0(N)})$ where $f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}$;

$$\Lambda_f := \{\int_\gamma \omega_f, \gamma \in H_1(X_0(N), \mathbb{Z})\},$$

and the map $\mathbb{C}/\Lambda_f \overset{\simeq}{\to} E(\mathbb{C})$ is the composition of the Weierstrass uniformization of the elliptic curve $E_f/\mathbb{Q}$ underlying the torus $\mathbb{C}/\Lambda_f$ with the choice of an isogeny $E_f \overset{\varphi}{\to} E$

defined over $\mathbb{Q}$; finally, c is the *Manin constant* defined by the relation $\pi^*(\omega_E) = c\omega_f$ where ω_E is the Néron differential of E, which is expected to be always $c = 1$.

This provides an efficient algorithm for computing Heegner points on elliptic curves $E/\mathbb{Q}$ of conductor N with respect to an imaginary quadratic field K in which all primes $p \mid N$ split. But as soon as we leave this scenario, $X_0^B(\mathfrak{N}^+)$ has no cusps and cuspforms in $S_2^B(\mathfrak{N}^+)$ admit no Fourier expansion: it seems hard to describe π explicitly in complex analytic terms.

As a sort of compensation, $X_0^B(\mathfrak{N}^+)$ and its automorphic forms *do admit* an explicit rigid analytic description at the primes $\mathfrak{p}$ dividing $\mathcal{D}_B \neq 1$, due to Tate, Cerednik, Drinfeld, Schneider and Teitelbaum. This allows for an analytic formula for Heegner points, as we now describe.

Let D_{an} denote the p-adic upper half-plane over $F_\wp$, the construction of which is recalled in e.g. §1 of [3.10] and [Dar04, Ch. V]; D_{an} is a rigid analytic variety over $F_\wp$ with

$$D_{\mathrm{an}}(\mathbb{C}_p) = \mathbb{C}_p \setminus F_\wp.$$

Let $D_{sp} := \mathcal{T}_\wp$ denote the tree of Bruhat and Tits whose set $\mathcal{V}$ of vertices is the set of maximal orders of $\mathbb{M}_2(F_\wp)$ and set $\mathcal{E}$ of oriented edges is the set of ordered pairs (v_1, v_2) with $v_1, v_2 \in \mathcal{V}$ such that $v_1 \cap v_2$ is an Eichler order of level $\wp$. We call $v_1 = s(e)$ and $v_2 = t(e)$ the source and the target of e, and write $\bar{e} = (v_2, v_1)$.

We denote by v_0 the vertex $\mathbb{M}_2(\mathcal{O}_{F_\wp})$ and by $\hat{v}_0$ the vertex

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), a, d \in \mathcal{O}_{F_\wp}, c \in \wp\mathcal{O}_{F_\wp}, b \in \wp^{-1}\mathcal{O}_{F_\wp} \right\}.$$

Set $e_0 = (v_0, \hat{v}_0)$.

There is a natural one-to-one correspondence between the set of ends of $\mathcal{T}_\wp$ and the boundary $\mathbf{P}^1(F_\wp)$ of $D_{\mathrm{an}}(\mathbb{C}_p)$; given an oriented edge $e \in \mathcal{E}$, the subset $U_e \subseteq \mathbf{P}^1(F_\wp)$ of ends leaving from e is open and compact, and $\{U_e\}_{e \in \mathcal{E}}$ forms a basis of the topology of $\mathbf{P}^1(F_\wp)$.

Assume $\mathcal{D}_B \neq 1$, fix a prime $\wp \mid \mathcal{D}_B$, let $K_\wp/F_\wp$ denote the quadratic unramified extension of $F_\wp$ and fix an embedding $K_\wp \hookrightarrow \mathbb{C}_p$. Then

- (Tate) There exists $q_f \in \mathcal{O}_{F_\wp}^\times$ for which there is an isomorphism of rigid analytic varieties $\Phi_{Tate} : \mathbb{G}_m/q_f^{\mathbb{Z}} \simeq E_f$ defined over $K_\wp$.

- (Cerednik-Drinfeld) For $\star = $ an, sp, there are isomorphisms

$$\Phi_{CD} : X_0^B(\mathfrak{N}^+)_\star \simeq B^{(\wp)\times} \backslash \big(D_\star \times [\prod_{\ell \neq \wp} B_\ell^\times / \prod_{\ell \neq \wp} (F_\ell^\times K_0^B(\mathfrak{N}^+)_\ell)]\big) \tag{3.12}$$

  where $X_0^B(\mathfrak{N}^+)_{\mathrm{an}}$ denotes the rigid analytic space associated with $X_0^B(\mathfrak{N}^+) \times F_\wp$ and $X_0^B(\mathfrak{N}^+)_{sp}$ is the dual graph of the special fiber of Morita's integral model of $X_0^B(\mathfrak{N}^+)$ over $\mathcal{O}_{F_\wp}$. These isomorphisms are defined over $K_\wp$ (resp. over its residue field) for $\star = $ an (resp. $\star = $ sp). As in (2.2),

$$X_0^B(\mathfrak{N}^+)_\star = \sqcup \Gamma_0^{(\wp)}(\mathfrak{N}^+, j) \backslash D_\star,$$

where $\Gamma_0^{(\wp)}(\mathfrak{N}^+, j)$ are defined as in (2.3): if we fix an Eichler $\mathcal{O}_F[1/\wp]$-order in $B^{(\wp)}$ and let $I_j^{(\wp)}$ run over the set of its left ideal classes, then $\Gamma_0^{(\wp)}(\mathfrak{N}^+, j)$ turns out to be the group of units of the right order $R_j^{(\wp)}$ of $I_j^{(\wp)}$ in $B^{(\wp)}$.

- Choose an ideal class $I^{(\wp)} \subset B^{(\wp)}$ corresponding to a connected component of the curve $X_0^B(\mathfrak{N}^+)_{\mathrm{an}}$, let $R^{(\wp)}$ denote its right order over $\mathcal{O}_F[1/\wp]$ and put $\Gamma^{(\wp)} = R^{(\wp)\times}$. An element $\tau \in \Gamma^{(\wp)} \backslash D_{\mathrm{an}}$ is a Heegner point if
$$\mathcal{O}_\tau^{(\wp)} := \{\gamma \in R^{(\wp)} : \gamma \cdot \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \lambda_\tau \begin{pmatrix} \tau \\ 1 \end{pmatrix} \text{ for some } \lambda_\tau \in \mathbb{C}_p\}$$
is an $\mathcal{O}_F[1/\wp]$-order in a totally complex quadratic extension K/F.

- (Schneider-Teitelbaum) There is an isomorphism
$$S_2^B(\mathfrak{N}^+, \psi = 1, \mathbb{C}_p) \simeq \oplus_j C_{\mathrm{har}}(\mathrm{Edges}(\mathcal{T}_\wp), \mathbb{C}_p)^{\Gamma_0^{(\wp)}(\mathfrak{N}^+, j)}$$
of Hecke modules, where $C_{har}(\mathcal{T}_\wp, \mathbb{C}_p)$ denotes the space of *harmonic cocycles*, namely maps
$$c : \mathrm{Edges}(\mathcal{T}_\wp) \to \mathbb{C}_p$$
such that
$$c(\bar{e}) = -c(e), \qquad \text{for any edge } e \text{ of } \mathcal{T}_\wp, \text{ and}$$
$$\sum_{s(e)=v} c(e) = 0, \qquad \text{for all vertices } v \text{ of } \mathcal{T}_\wp.$$

Let $c_f \in C_{har}(\mathrm{Edges}(\mathcal{T}_\wp), \mathbb{Z})^{\Gamma^{(\wp)}}$ be the harmonic cocyle associated with the Hilbert modular form f and the chosen connected component of $X_0^B(\mathfrak{N}^+)_{\mathrm{an}}$. Over $\mathbb{C}_p$, the modular parametrisation (3.7) becomes the map
$$\pi : \mathrm{Div}_0(\Gamma^{(\wp)} \backslash D_{\mathrm{an}}) \longrightarrow \mathbb{C}_p^\times / q_f^{\mathbb{Z}} \stackrel{\Phi_{Tate}}{\longrightarrow} E_f(\mathbb{C}_p) \stackrel{\varphi}{\longrightarrow} E(\mathbb{C}_p) \tag{3.13}$$
sencing the degree 0 divisor $\tau - \tau'$ on $\mathcal{H}_p$ to the "multiplicative integral"
$$⨰_\tau^{\tau'} f, \tag{3.14}$$
where
$$⨰_\tau^{\tau'} f = ⨰_{\mathbb{P}^1(F_\wp)} (\frac{t - \tau'}{t - \tau}) dc_f(t) := \lim_{n \to \infty} \prod_{\mathrm{dist}(e, e_0) = n} (\frac{t_e - \tau'}{t_e - \tau})^{c_f(e)}. \tag{3.15}$$
Here $t_e \in U_e$ is any sample point.

A fundamental idea of Pollack and Stevens, based on Stevens' theory of *overconvergent modular symbols*, leads to an efficient algorithm for calculating the integrals of (3.15) in practice. Subsequent improvements by M. Greenberg make it easy to implement in a wide variety of settings. See [DP], or Robert Pollack's lectures at this Winter School, for details.

We are thus able to constructibly answer Question 2.1 in a number of situations which, nonetheless, represents a poor understanding of the question in full generality. In particular, the following questions (arranged in decreasing order of generality and difficulty) remain completely open:

1. What if $\mathrm{ord}_{s=1} L(E/K, s) > 1$?

2. Even if $\mathrm{ord}_{s=1} L(E/K, s) = 1$, what if K is not a totally complex quadratic extension of F?

3. Even if K is totally complex, what if hypothesis (JL) does not hold?

4. Even if hypothesis (JL) holds, what if $[F : \mathbb{Q}] > 1$ is odd and $\mathcal{D}_B = 1$? In that case a Heegner point construction is available, but the absence of a prime at which the relevant Shimura curve admits an explicit rigid analytic description à la Cerednik-Drinfeld means that we have no good numerical algorithm for computing these Heegner points in practice.

References

For the classical theory of complex multiplication on elliptic curves and abelian varieties, and the connection with Heegner points on modular curves, the reader is urged to consult any of the many excellent manuscripts available in the literature, like

3.1. Birch, B., *Heegner points: the beginnings*, in H. Darmon, S. Zhang, *Heegner Points and Rankin L-Series*, Mathematical Sciences Research Institute Publications **49**, Cambridge University Press.

3.2. D. A. Cox. *Primes of the form $x^2 + ny^2$: Fermat, class field theory, and complex multiplication.* John Wiley & Sons (1989).

3.3. B. H. Gross, *Heegner points on $X_0(N)$*, in *Modular forms*, edited by R. A. Rankin, Chichester, Ellis Horwood (1984), 87–105.

3.4. G. Shimura, Y. Taniyama, *Complex Multiplication of Abelian Varieties and Its Applications to Number Theory*, Tokyo: Mathematical Society of Japan, 1961.

3.5. J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Grad. texts Math. **151**, Springer Verlag 1991.

For an account of the theory of Heegner points on Shimura curves associated to non-split quaternion algebras, the reader may consult [BD98] and

3.6. M. Greenberg, *Heegner points and rigid analytic modular forms*, McGill Ph. D thesis, 2006.

3.7. S. Molina, *Ribet bimodules and the specialization of Heegner points*, to appear in Israel J. Math.

3.8. S. Zhang. *Heights of Heegner points on Shimura curves*, Ann. of Math. (2) **153** (2001), no. 1, 27–147.

Finally, for background on the p-adic upper half-plane and the rigid analytic uniformization of Shimura curves, we again can suggest [BD98] and [3.6], together with [Dar04] and

3.9. J.-F. Boutot, H. Carayol. *Uniformisation p-adique des courbes de Shimura: les théorèmes de Čerednik et de Drinfeld*, Astérisque **196-197** (1991), no. 7, 45–158.

3.10. S. Dasgupta, J. Teitelbaum, *The p-adic upper half plane*, in p-adic Geometry: Lectures from the 2007 Arizona Winter School, ed. D. Savitt, D. Thakur. University Lecture Series **45**, Amer. Math. Soc., Providence, RI, 2008.

Chapter 4

Algebraic cycles and de Rham cohomology

4.1 Algebraic cycles

This chapter is devoted to review the basic rudiments of the classical theory of algebraic cycles and various cohomology theories on algebraic varieties, with particular emphasis on the algebraic de Rham cohomology and its classical complex incarnations.

In order to motivate the introduction of this material at this point of the notes, let us invoke again, in a slightly different guise, the modular parametrization (3.7) that all elliptic curves E over $\mathbb{Q}$ are equipped with, thanks to the work of Wiles; let N denote the conductor of E and let X denote the modular curve $X_0(N)$ of level N.

Fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$; for any field extension $K/\mathbb{Q}$ in $\bar{\mathbb{Q}}$ let $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ and let

$$\mathrm{CH}^1(X)(K) := \mathrm{Pic}(X)(K) = (\mathrm{Div}(X)(\bar{\mathbb{Q}})/\sim_{rat})^{G_K}$$

denote the module of rational equivalence classes of divisors on the curve X_K. Define

$$\deg : \mathrm{CH}^1(X)(\bar{\mathbb{Q}}) \longrightarrow \mathbb{Z}, \quad [\sum n_k P_k] \mapsto \sum n_k \tag{4.1}$$

and

$$\mathrm{CH}^1(X)_0(K) = \mathrm{Pic}(X)_0(K) = \ker(\deg_{|\,\mathrm{CH}^1(X)(K)}),$$

the subgroup of divisors of degree zero on X, which in the language employed in this chapter will also be referred to as the subgroup of rational equivalence classes of null-homologous cocycles of codimension 1 on X_K. As is well-known, these are not mere groups but the group of K-rational points of an algebraic group defined over $\mathbb{Q}$ which we might still denote by $\mathrm{CH}^1(X)_0$ by a slight abuse of notation. It is called the jacobian variety of X. Equation (3.7) says that there exists a morphism of algebraic groups

$$\pi : \mathrm{CH}^1(X)_0 \longrightarrow E \tag{4.2}$$

defined over $\mathbb{Q}$, inducing a homomorphism of abelian groups $\mathrm{CH}^1(X)_0(K)\longrightarrow E(K)$ for each $K/\mathbb{Q}$. In the previous chapter this was used as a machine for producing (Heegner) points on E. Lacking a direct construction of points on $X(K)$, we were instead able to produce a natural supply of points in $\mathrm{CH}^1(X)_0(K)$ which can be mapped down to $E(K)$ via (4.2).

Can one mimic the same picture by replacing X by other (possibly higher-dimensional) varieties? Chapter 5 will report on the attempt made in [BDP2] to bring to practice this (so far vaguely formulated) suggestion in the very special scenario where X is a specific family of abelian varieties fibered over a modular curve, and E is a suitable elliptic curve *with complex multiplication*. Chapter 7, which is part of an ongoing project of the authors with Ignacio Sols, treats a somewhat more gneral setting where V is fibered over a product of three modular curves and E is a general elliptic curve over $\mathbb{Q}$, not necessarily with complex multiplication. Before delving into these concrete settings, we need to introduce some of the basic ingredients that are needed along the way.

Let F be a field of characteristic 0 and fix an algebraic closure $\bar{F}$ of F. Let V be a nonsingular irreducible projective algebraic variety defined over F of dimension $d \geq 1$ and put $V_{\bar{F}} = V \times_{\mathrm{Spec}(F)} \mathrm{Spec}(\bar{F})$.

Choose an integer $1 \leq c \leq d$. An algebraic cycle of codimension c on V is a formal sum $Z = \sum n_k Z_k$, where each Z_k is an irreducible not necessarily non-singular sub-variety of codimension c in $V_{\bar{F}}$, and $n_k \in \mathbb{Z}$. Two codimension c cycles Z and Z' are said to be *rationally equivalent*, denoted $Z \sim_{rat} Z'$, if there exists a finite collection of sub-varieties $\{V_i\}$ of $V_{\bar{F}}$ together with rational functions $f_i \in \bar{F}(V_{\bar{F}})$ such that

$$Z' - Z = \sum \mathrm{div}(f_i).$$

Write $[Z]$ for the class of Z up to rational equivalence; for any field extension K/F in $\bar{F}$, define $\mathrm{CH}^c(V)(K)$ to be the group of rational equivalence classes of codimension c cycles Z on V such that ${}^\sigma Z \sim_{rat} Z$ for all $\sigma \in G_K$.

How should the degree map (4.1) generalize to the context of cycles on higher dimensional varieties? Note that, although there is an obvious mapping

$$Z = \sum n_k Z_k \mapsto \deg(Z) := \sum n_k \in \mathbb{Z},$$

in general it is not true that $\deg(Z) = \deg(Z')$ for any two rationally equivalent codimension c cycles on V. This does hold true when $c = d$: a codimension d cycle on V is just a formal sum of points, and the difference of two rationally equivalent codimension d cycles is the sum of the divisors of a collection of rational functions on curves V_i embedded in V. Since the degree of a rational function on a curve is 0, it follows that $\deg(Z) = \deg(Z')$. Hence the degree function descends to $\mathrm{CH}^d(V)$, inducing a well-defined map

$$\deg : \mathrm{CH}^d(V)(\bar{\mathbb{Q}})\longrightarrow\mathbb{Z}. \tag{4.3}$$

As above, one defines $\mathrm{CH}^d(V)_0$ as the kernel of deg.

For $c < d$, however, there exist rationally equivalent cycles of different "degree: in the above naive sense. More precisely, fix an embedding $V \hookrightarrow \mathbf{P}^N$ of V into a projective space.

For each $n \geq 1$, there exists a subvariety $W_n \subset \mathbf{P}^N$ of codimension c and degree n^c in $\mathbf{P}^N$ such that $V \cap W_n$ is irreducible and the cycles $\frac{1}{n^c} \cdot V \cap W_n$ are all equivalent in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{CH}^c(V)$. Indeed, this follows by recursively applying the theorem of Bertini c times (cf. e.g. Ch. II, §8 of [4.7]) to the variety V embedded in the projective space of degree n forms in $\mathbf{P}^N$ via the n-th Veronese embedding. Note for example that

$$Z_1 = 2^c \cdot V \cap W_1 \sim_{rat} Z_2 = V \cap W_2, \quad \text{and} \quad \deg(Z_1) = 2^c \quad \text{while} \quad \deg(Z_2) = 1,$$

hence the degree mapping does not provide a well-defined homomorphism on $\mathrm{CH}^c(V)$. The correct generalization of (4.3) is called the *cycle class map*, which admits various cohomological versions (one for each of the Weil cohomology theories $V \mapsto H^*(V)$ which are available for V) and corresponds to a homomorphism

$$\mathrm{cl} : \mathrm{CH}^c(V) \longrightarrow H^{2c}(V)(c), \tag{4.4}$$

with values on (a twist of) the $2c$-th cohomology group of V.

4.2 de Rham cohomology

Here we shall review and use the de Rham cohomology of V in its various avatars. We refer the reader to [4.8] for the étale cohomology theory and the corresponding version of (4.4).

Let $\mathcal{O}_V$ denote the structure sheaf of regular functions on V and let Ω^k_V denote the sheaf of regular differential k-forms on V. The algebraic de Rham cohomology $H^*(V)$ of V is defined to be the hypercohomology of the de Rham complex

$$0 \to \mathcal{O}_V \xrightarrow{d_0} \Omega^1_V \xrightarrow{d_1} \Omega^2_V \to \cdots \to \Omega^d_V \to 0. \tag{4.5}$$

For $k \geq 0$, $H^k(V)$ is an F-vector space of finite dimension, which is trivial if $k > 2n$ and can be introduced explicitly as follows. Choose a covering $\mathcal{U} = \{U_i\}_{i \in I}$ of V by Zariski open subsets indexed by a countable set I and let

$$0 \to \Omega^p_V \to C^0(\Omega^p_V) \to \cdots \to C^q(\Omega^p_V) \xrightarrow{c_q^{(p)}} C^{q+1}(\Omega^p_V) \to \cdots$$

be the Cech resolution of Ω^p_V with respect to $\mathcal{U}$. The exterior derivatives d_p induce differentials

$$d_{p,q} : C^q(\Omega^p_V) \longrightarrow C^q(\Omega^{p+1}_V),$$

and the total complex $(\mathrm{Tot}^*(C^*(\Omega^*_V)), \delta)$ of $C^*(\Omega^*_V)$ is defined by setting

$$\mathrm{Tot}^k_V = \mathrm{Tot}^k(C^*(\Omega^*_V)) = \oplus_{p+q=k} C^q(\Omega^p_V), \qquad \delta_t = \sum_{p+q=k} d_{p,q} + (-1)^p c_p^{(q)} : \mathrm{Tot}^k_V \to \mathrm{Tot}^{k+1}_V.$$

The k-th algebraic de Rham cohomology group of V is defined by

$$H^k_{\mathrm{dR}}(V/F) := H^k(\mathrm{Tot}^*(C^*(\Omega^*_V))) = \ker(\delta_k)/\mathrm{Im}(\delta_{k-1}). \tag{4.6}$$

Occasionally we will write $H^k_{\mathrm{dR}}(V)$ for $H^k(_{\mathrm{dR}}(V/F)$ when the field F can be inferred from the context.

It can be shown that $H^k_{\mathrm{dR}}(V/F)$ is a finite-dimensional F-vector space with dimension equal to the k-th Betti number of $V(\mathbb{C})$. Another fundamental property of the algebraic de Rham cohomology groups that we shall be recurrently using is the Künneth formula, which asserts that if $V = V_1 \times V_2$ is the product of two varieties, then the pull-back under the two natural projections $p_i : V \longrightarrow V_i$ induces an isomorphism

$$(p_1^*, p_2^*) : \oplus_{p+q=k} H^p_{\mathrm{dR}}(V_1) \otimes_F H^q(V_2) \xrightarrow{\sim} H^k_{\mathrm{dR}}(V). \tag{4.7}$$

This is in fact a very particular instance of the sort of calculation one can make by means of the Leray-Serre spectral sequence of the cohomology of a variety V equipped with a flat fibration

$$\pi : V \longrightarrow B$$

over a smooth base B with complete, generically smooth, and topologically equivalent fibres (see [4.5] for more details).

The total complex is naturally equipped with the filtration

$$F^i \mathrm{Tot}^k_V := \oplus_{p+q=k, p\geq i} C^q(\Omega^p),$$

which induces a decreasing, exhaustive and separated filtration

$$F^0 H^k_{\mathrm{dR}}(V) = H^k_{\mathrm{dR}}(V) \supseteq F^1 H^k_{\mathrm{dR}}(V) \supseteq \cdots \supseteq \{0\}$$

on $H^k_{\mathrm{dR}}(V)$, called the *Hodge filtration.*

For each $r \in \mathbb{Z}$, the r-th Tate twist of $H^k_{\mathrm{dR}}(V)$ is defined to be

$$H^k_{\mathrm{dR}}(V)(-r) = H^k_{\mathrm{dR}}(V) \otimes_F H^2_{\mathrm{dR}}(\mathbf{P}^1/F)^{\otimes r},$$

where we adopt the usual convention that $S^{\otimes(-1)} := \mathrm{Hom}(S, F)$ for an F-vector space S. Since $\dim_F H^2_{\mathrm{dR}}(\mathbf{P}^1/F) = 1$, the underlying vector spaces of $H^k_{\mathrm{dR}}(V)$ and $H^k_{\mathrm{dR}}(V)(r)$ are abstractly isomorphic, but the respective filtrations are shifted r positions one from another.

By a fundamental theorem (proved in much greater generality by Deligne and Illusie in [4.3]), the spectral sequence

$$E_1^{p,q} = H^q(V, \Omega^p_V) \Rightarrow H^{p+q}_{\mathrm{dR}}(V)$$

associated with the filtered complex (Tot^*_V, F^*) degenerates at the E_1 term and as a consequence there is in particular a short exact sequence

$$0 \longrightarrow H^0(V, \Omega^1_V) \longrightarrow H^1_{\mathrm{dR}}(V) \longrightarrow H^1(V, \mathcal{O}_V) \longrightarrow 0. \tag{4.8}$$

The algebraic de Rham cohomology groups are also equipped, for each $0 \leq k \leq 2d$, with an alternate non-degenerate pairing

$$\langle \, , \rangle : H^k_{\mathrm{dR}}(V) \times H^{2d-k}_{\mathrm{dR}}(V)(d) \longrightarrow H^{2d}_{\mathrm{dR}}(V)(d) \simeq F, \tag{4.9}$$

called the *Poincaré pairing*, from which it follows that $H^{2d-k}_{\mathrm{dR}}(V)(d) \simeq H^k_{\mathrm{dR}}(V)^\vee$. It follows from (4.5) and (4.9) for $k=0$ and $2d$ that

$$H^0_{\mathrm{dR}}(V) = \ker(H^0(V,\mathcal{O}_V) \longrightarrow H^0(V,\Omega^1_V)) \simeq F \simeq H^{2d}_{\mathrm{dR}}(V)^\vee.$$

For $k \geq 1$ one can likewise give an explicit description of $H^k_{\mathrm{dR}}(V)$ by picking an affine covering of V and unwinding (4.5). We content ourselves here with the details in the case of the first cohomology group of a curve, whose description in terms of hypercocycles is relatively simple because one can always work with the affine covering

$$\mathcal{U} = \{U_1 = V \setminus \{P_1\}, U_2 = V \setminus \{P_2\}\}$$

of the curve V for any choice of points $P_1, P_2 \in V$. An easy exercise that is left to the reader then shows that

$$H^1_{\mathrm{dR}}(V) = \frac{\mathbb{Z}^1(V;U)}{\mathbb{B}^1(V;U)}, \tag{4.10}$$

where

$$\mathbb{Z}^1(V;U) := \{(\omega_1,\omega_2,F_{12}) \in \Omega^1(U_1) \times \Omega^1(U_2) \times \mathcal{O}(U_1 \cap U_2) : (\omega_1 - \omega_2)_{|U_1 \cap U_2} = dF_{12}\}$$

is the group of 1-hypercocycles on V attached to the affine covering U, and

$$\mathbb{B}^1(V;U) := \{(df_1, df_2, (f_1 - f_2)_{|U_1 \cap U_2}),\ f_1 \in \mathcal{O}(U_1), f_2 \in \mathcal{O}(U_2)\}.$$

is the group of hyper 1-coboundaries attached to this covering.

Below we shall explain alternative, more classical and pleasant descriptions of some of these spaces. We are nevertheless specially interested in understanding the case $k=1$, not only because this completes the whole picture when V is a curve, but also because all higher-dimensional varieties we shall be considering are naturally given as fibrations $\pi : V \longrightarrow B$ over a one-dimensional base B whose fibers are again (products of) curves; for such varieties, (4.7) and the Leray-Serre spectral sequence reduces the computation of all cohomology groups of V to the case $k=1$.

It follows from (4.8) that $H^1_{\mathrm{dR}}(V)$ contains the space of global regular differentials 1-forms on V. It is thus natural to ask whether it is possible to give a description of the whole space $H^1_{\mathrm{dR}}(V)$ in terms of global differentials on V. As we shall see, this is afforded by the classical notion of differential forms of the *second kind*.

Let $F(V)$ denote the function field of V over F and $\Omega^1_{F(V)/F}$ denote the sheaf of rational differentials on V. Let $\eta \in H^0(V,\Omega^1_{F(V)/F})$ be a global section of $\Omega^1_{F(V)/F}$; for any irreducible effective divisor D on V, let $\mathcal{O}_{V,D}$ denote the stalk of V at D, which is a discrete valuation ring. Choose an uniformizer π of $\mathcal{O}_{V,D}$; we may write the local expression of η about D as $\eta_D = \sum_{n \geq N_D} a_n \pi^n$ for some $N_D \in \mathbb{Z}$ and $a_n \in F$. The integer N_D and the element $a_{-1} \in F$ are both independent of the choice of π and are called, respectively, the valuation and the residue of η about D.

Definition 4.1. Let $\eta \in H^0(V, \Omega^1_{F(V)/F})$ be a rational differential 1-form on V. We say η is of the *first* (respectively, *second*) *kind* if $N_D \geq 0$ (resp. $\mathrm{res}_D(\eta) = 0$) for all irreducible effective divisor D on V.

Write $\Omega^1(V) = H^0(V, \Omega^1_V)$ and $\Omega^1_{II}(V)$ for the spaces of differential forms of the first and second kind on V. Note that $\Omega^1_{II}(V)$ contains both $\Omega^1(V)$ and $dF(V)$. Furthermore, if $(\omega_1, \omega_2, F_{12}) \in \mathbb{Z}^1(V; U)$ is a one-hypercocycle attached to U, the regular differential ω_1 on U_1 extends to a differential of the second kind on V, since it is regular outside P_1 and its residue at P_1 is

$$\mathrm{res}_{P_1}(\omega_1) = \mathrm{res}_{P_1}(\omega_2 + dF_{12}) = 0.$$

Theorem 4.2. *[4.1], [4.6]. There is a canonical isomorphism*

$$H^1_{\mathrm{dR}}(V) \simeq \Omega^1_{II}(V)/dF(V). \tag{4.11}$$

If $d = \dim(V) = 1$, it is given by the rule

$$(\omega_1, \omega_2, F_{12}) \mapsto [\omega_1]. \tag{4.12}$$

When V is a curve, the Poincaré pairing induces a pairing on $H^1_{II}(V)$ via (4.11) which is explicitly given by the rule

$$\langle \eta, \eta' \rangle = \sum_x \mathrm{res}_x(\tilde{\eta}_x \cdot \eta') \tag{4.13}$$

where $\tilde{\eta}_x \in \mathcal{O}_{V,x}$ is a local primitive of $\eta_x \in \Omega^1_{\mathcal{O}_{V,x}/F}$. This expression is well-defined thanks to the residue theorem, which asserts that

$$\sum_{x \in V} \mathrm{res}_x(\eta) = 0 \quad \text{for all } \eta \in H^0(V, \Omega^1_{F(V)/F}).$$

(Cf. Ch. III.7.14.2 of [4.7]).

We strongly encourage the reader who is not familiar with these notions to write down the details of the proof that the map in Theorem 4.2 and the formula in (4.13) are well-defined.

Remark 4.3. In view of Theorem 4.2 above, one may wonder whether there exists a reasonable notion of differential k-forms of the second kind when $k \geq 2$, providing a more workable description of $H^k_{\mathrm{dR}}(V_{\mathbb{C}})$. In an article of great historical importance [4.1], Atiyah and Hodge introduce such forms as those global meromorphic differential k-forms η on V such that for all sufficiently large divisors $D \subset V$ on the complement of which η is regular,

$$\int_\gamma \eta = 0, \text{ for all } \gamma \in H_k(V \setminus D, \mathbb{Z}) \text{ homologically trivial in } V.$$

Write $\Omega^k_{II}(V)$ for the space of differential k-forms of the second kind on V, which again contains $d\Omega^{k-1}_{F(V)/F}$, and define

$$H^k_{II}(V) = \Omega^k_{II}(V)/d\Omega^{k-1}_{F(V)/F}.$$

For $k=1$, Theorem 4.2 asserts that $H^1_{\mathrm{dR}}(V) \simeq H^1_{II}(V)$, while for $k \geq 2$ the geometry of V may impose some obstruction for the spaces $H^k_{\mathrm{dR}}(V)$ and $H^k_{II}(V)$ to be isomorphic. For $k=2$, for instance, we recover the so-called transcendental part

$$H^2_{\mathrm{dR}}(V_{\mathbb{C}})_{tr} := \frac{H^2_{\mathrm{dR}}(V_{\mathbb{C}})}{\mathrm{cl}(\mathrm{CH}^1(V)) \otimes \mathbb{C}} \simeq H^2_{II}(V_{\mathbb{C}}).$$

Since V is of finite type over F, we may assume without loss of generality that F is finitely generated over $\mathbb{Q}$ and we can fix an embedding $F \hookrightarrow \mathbb{C}$ of F into the field of complex numbers. The set of complex points of $V_{\mathbb{C}}$ endowed with the euclidean topology has a natural structure of complex analytic variety, which we may denote by $V_{\mathbb{C}}^{\mathrm{an}}$. The algebraic de Rham cohomology groups $H^k_{\mathrm{dR}}(V_{\mathbb{C}})$ that we just introduced admit at least two other, more classical descriptions. Namely, let

$$H^k_{\mathrm{B}}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}) = \mathrm{Hom}(H_k(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}), \mathbb{Z})$$

denote the finitely generated abelian group of *singular* or *Betti cohomology* of the topological manifold underlying $V_{\mathbb{C}}^{\mathrm{an}}$, as defined in any standard treatment of algebraic topology. Also, if we let $A^k(V)$ denote the real vector space of $\mathbb{R}$-valued smooth differential k-forms on the real differentiable manifold underlying $V_{\mathbb{C}}^{\mathrm{an}}$ and let $d_{\mathbb{R},k} : A^k(V) \to A^{k+1}(V)$ denote the usual exterior differential, the quotient space

$$H^k_{\mathrm{dR}}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{C}) = H^k_{\mathrm{dR}}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \ker(d_{\mathbb{R},k} \otimes \mathbb{C}) / \mathrm{Im}(d_{\mathbb{R},k-1} \otimes \mathbb{C})$$

of complex-valued *closed* forms up to *exact* forms is called the (classical, complex) k-th de Rham cohomology group of $V_{\mathbb{C}}^{\mathrm{an}}$.

Theorem 4.4. *There are functorial isomorphisms*

$$H^k_{\mathrm{B}}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^k_{\mathrm{dR}}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq H^k_{\mathrm{dR}}(V_{\mathbb{C}}).$$

Notice these isomorphisms provide a canonical integral (respectively, real) structure on the complex vector space $H^k_{\mathrm{dR}}(V_{\mathbb{C}})$. In particular there is a natural action $\eta \mapsto \bar{\eta}$ of complex conjugation on $H^k_{\mathrm{dR}}(V_{\mathbb{C}})$. By means of the above isomorphism, the Poincaré pairing on $H^k_{\mathrm{dR}}(V_{\mathbb{C}})$ is simply the complexification of the usual integer-valued intersection pairing

$$\langle\, , \rangle : H^k_{\mathrm{B}}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}) \times H^{2d-k}_{\mathrm{B}}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}) \longrightarrow \mathbb{Z}, \quad (Z_1, Z_2) \mapsto Z_1 \cdot Z_2,$$

which in turn translates into the context of smooth differential forms on $V_{\mathbb{C}}^{\mathrm{an}}$ as

$$\langle\, , \rangle : H^k_{\mathrm{dR}}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{R}) \times H^{2d-k}_{\mathrm{dR}}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{R}) \longrightarrow \mathbb{R}, \quad \langle \omega_1, \omega_2 \rangle = \frac{1}{(2\pi i)^n} \int_{V(\mathbb{C})} \omega_1 \wedge \omega_2.$$

The main theorem of Hodge theory for complete non-singular complex manifolds asserts that every cohomology class in $H^k_{\mathrm{dR}}(V_{\mathbb{C}})$ admits a unique harmonic representative. Letting

$H^{p,q}(V_{\mathbb{C}})$ denote the subspace of classes in $H^k_{\mathrm{dR}}(V_{\mathbb{C}})$ represented by a harmonic form of type (p,q), we obtain the fundamental *Hodge decomposition* of the complex de Rham cohomology:

$$H^k_{\mathrm{dR}}(V_{\mathbb{C}}) \simeq \oplus_{p+q=k} H^{p,q}(V_{\mathbb{C}}) \tag{4.14}$$

It can be shows that

$$H^{p,q}(V_{\mathbb{C}}) \simeq H^q(V^{\mathrm{an}}_{\mathbb{C}}, \Omega^p_{V_{\mathbb{C}}})$$

is isomorphic (as a complex vector space) to the successive graded pieces in the Hodge filtration of $H^k_{\mathrm{dR}}(V/\mathbb{C})$, that $H^{p,q}(V_{\mathbb{C}}) = \overline{H^{q,p}(V_{\mathbb{C}})}$, and that

$$F^i H^k_{\mathrm{dR}}(V_{\mathbb{C}}) \simeq \oplus_{p+q=k, p\geq i} H^{p,q}(V_{\mathbb{C}}).$$

(Cf. e.g. Ch. 0, §7 of [4.6] or Ch. 5 of [4.10].)

4.3 Cycle classes and Abel-Jacobi maps

Having completed our digression on the de Rham cohomology groups of V, we now resume the discussion of cycles on V and the cycle class map (4.4) initiated at the beginning of the chapter, In order to define (4.4) with values on $H^{2c}_{\mathrm{dR}}(V)(c)$, one usually starts by defining the class of an irreducible subvariety Z of codimension c in V as the c-th Chern class of the structure sheaf $\mathcal{O}_Z$ of Z, regarded as a coherent sheaf on V, and extends the definition of (4.4) to the whole group $\mathrm{CH}^c(V)$ by linearity.

Alternatively, one may further assume that Z is non-singular and use the Gysin sequence attached to the pair $(V, V \setminus Z)$ to define $\mathrm{cl}(Z)$ as the image of 1 under the map

$$H^0(Z) \xrightarrow{\sim} H^{2c}_Z(V)(c) \longrightarrow H^{2c}(V)(c).$$

Over the complex numbers, $\mathrm{cl}(Z) \in H^{2c}_{\mathrm{dR}}(V_{\mathbb{C}})(c)$ is simply the image of the homology class $[Z] \in H_{2d-2c}(V^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})$ under the isomorphism

$$H_{2d-2c}(V^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C}) \simeq H^{2c}_{\mathrm{B}}(V^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C})(c) \simeq H^{2c}_{\mathrm{dR}}(V_{\mathbb{C}})(c)$$

provided by Poincaré duality and Theorem 4.4. One then extends this definition to singular cycles by standard methods. The reader is encouraged to remedy this inadequate treatment of the foundations by consulting the details that are explained in [4.4], [4.8] or [4.10].

The subgroup of null-homologous cycles of codimension c on V is defined to be

$$\mathrm{CH}^c(V)_0 = \ker(\mathrm{cl}).$$

It is independent of the chosen cohomology theory. When $d = \dim(V) = 1$, it is well-known that there is an isomorphism

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^c(V)_0(\mathbb{C}) \xrightarrow{\sim} H^{1,0}(V_{\mathbb{C}})^{\vee}/H_1(V^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z}), \quad D = [Q-P] \mapsto \int_P^Q, \tag{4.15}$$

where

$$\int_P^Q : H^{1,0}(V_{\mathbb{C}}) = H^0(V_{\mathbb{C}}^{\mathrm{an}}, \Omega^1) \longrightarrow \mathbb{C}$$

is the linear functional defined by mapping a holomorphic differential 1-form ω on the Riemann surface $V_{\mathbb{C}}^{\mathrm{an}}$ to the complex line integral of ω along any path joining the points P and Q.

For varieties of dimension $d > 1$ and for any $c \leq d$, one defines the c-th intermediate jacobian of V as the (typically non algebraic) torus

$$J^c(V) = \frac{\mathrm{F}^c H_{\mathrm{dR}}^{2c-1}(V_{\mathbb{C}}^{\mathrm{an}})^\vee}{H_{2c-1}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z})}. \tag{4.16}$$

The complex Abel-Jacobi map is given by

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^c(V)_0(\mathbb{C}) \longrightarrow J^c(V), \qquad \Delta \mapsto \int_{\partial^{-1}\Delta} \tag{4.17}$$

where $\tilde{\Delta} = \partial^{-1}\Delta$ is a $2(d-c)+1$ real dimensional piecewise differentiable chain on the real manifold $V(\mathbb{C})$ with boundary Δ. In general (4.17) fails to be an isomorphism, but if F is a number field then Bloch and Kato ([4.2], [4.9]) conjecture that it is injective.

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Chapter 5

Chow-Heegner points

The aim of this chapter is to develop in more detail the suggestion sketched in the introduction of the previous one of generalizing the classical modular parametrizations (3.7) of elliptic curves to higher-dimensional scenarios, by exploiting the Abel-Jacobi map on null-homologous algebraic cycles described in the previous chapter.

5.1 Generalities

Let E/F be an elliptic curve defined over a number field F. As in Chapter 4, let V/F be a non-singular projective algebraic variety of dimension $d \geq 1$. Assume that there exists a cycle

$$\Pi \in \mathrm{CH}^{d+1-c}(V \times E)(F)$$

of codimension $d+1-c$ on the product of V by E for some $c \leq d$. Letting

$$\pi_V : V \times E \to V, \qquad \pi_E : V \times E \to E$$

denote the natural projections onto the two factors, the rule

$$\Delta \ \mapsto \ P_\Delta := \pi_{E,*}(\pi_V^* \Delta \cdot \Pi) \tag{5.1}$$

induces a homomorphism

$$\pi : \mathrm{CH}^c(V)_0(\bar{F}) \longrightarrow \mathrm{CH}^1(E)_0(\bar{F}) = E(\bar{F}), \tag{5.2}$$

which we may call the modular parametrization of E associated to the pair (V, Π).

Since Π is rational over F, the map π of (5.2) is in fact a morphism of G_F-modules. In particular, if K/F is any finite extension (over which we wish to construct rational points on E), taking invariants under the open subgroup $G_K \subseteq G_F$, the map π_Π restricts to a homomorphism

$$\pi : \mathrm{CH}^c(V)_0(K) \longrightarrow \mathrm{CH}^1(E)_0(K) = E(K). \tag{5.3}$$

Together with (5.2), it is often interesting (and practical, if one wishes to perform numerical computations) to dispose of explicit archimedean descriptions, either complex or rigid analytic, of the modular parametrization π. By way of illustration, in Chapter 3 this was worked out for the classical modular parametrization (3.7) in (3.11). The reader is encouraged to compare those descriptions with the one below.

After fixing an embedding $F \hookrightarrow \mathbb{C}$, the Abel-Jacobi map (4.17) gives rise to a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^c(V)_0(\mathbb{C}) \arrow[r, "\mathrm{AJ}_{\mathbb{C}}"] \arrow[d, "\pi"'] & J^c(V_{\mathbb{C}}^{\mathrm{an}}) \arrow[d, "\pi_{\mathbb{C}}"] \\ E(\mathbb{C}) \arrow[r, "\mathrm{AJ}_{\mathbb{C}}"] & \mathbb{C}/\Lambda_E, \end{tikzcd} \tag{5.4}$$

which should be regarded as the complex analytic manifestation of the global morphism π. On the right hand of the diagram,

- $J^c(V_{\mathbb{C}}^{\mathrm{an}})$ is the c-th intermediate jacobian of V introduced in (4.16);
- Λ_E is the lattice of periods of E against a holomorphic Néron differential ω_E on $E_{\mathbb{C}}^{\mathrm{an}}$:

  $$\Lambda_E := \{\int_\gamma \omega_E, \gamma \in H_1(E_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z})\}.$$

  The rule $\varphi \mapsto \varphi(\omega_E)$ yields an isomorphism $H^{1,0}(V_{\mathbb{C}})^\vee / H_1(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}) \simeq \mathbb{C}/\Lambda_E$ of analytic tori.
- The map $\pi_{\mathbb{C}} : J^c(V_{\mathbb{C}}^{\mathrm{an}}) \longrightarrow \mathbb{C}/\Lambda_E$ sends a functional ψ on $\mathrm{F}^c H_{dR}^{2c-1}(V_{\mathbb{C}}^{\mathrm{an}})$ to

  $$\pi_{\mathbb{C}}(\psi) := \psi(\pi_E^*(\omega_E)) \in \mathbb{C}/\Lambda_E. \tag{5.5}$$

Naturally enough, one would also like to impose some condition on the pair (V, Π) to ensure that the maps π of (5.2) and $\pi_{\mathbb{C}}$ of (5.4) are non-trivial. In light of (5.5), it is reasonable to require that

$$\pi_{V,*}\pi_E^* : H_{dR}^1(E) \longrightarrow H_{dR}^{2d-2c+1}(V) \tag{5.6}$$

be non-zero on $H^0(E, \Omega_E^1)$.

Of course, the modular parametrization (5.3) can only be useful when the variety V is equipped with an interesting supply of null-homologous cycles

$$\Delta \in \mathrm{CH}^c(V)_0(K).$$

In the setting of curves discussed in Chapter 3, these cycles where just the Heegner divisors

$$D = ([\tau] - \infty) \in \mathrm{CH}^1(X_0(N))_0(H_{\mathcal{O}_\tau}).$$

In higher dimension, it is well-known that Shimura varieties associated to a reductive group $G_{/\mathbb{Q}}$ host special cycles defined over well-understood number fields; exploiting these cycles to construct non-trivial algebraic points on elliptic curves is doubtlessly a promising line of research. The notion of what it means for an algebraic cycle Δ to be *interesting* has been kept deliberately vague. The *ne plus ultra* would be a collection $\{\Delta_1, \ldots, \Delta_r\} \subset \mathrm{CH}^c(V)_0(K)$ of such cycles such that $\mathbb{Q} \otimes E(K) = \oplus_{i=1}^r \mathbb{Q} \cdot P_{\Delta_i}$, where $r = \operatorname{rank} E(K)$. Producing such collections seems (at present) like a tall order, particularly when $r > 1$. (One might speculate that this is a natural way to approach the Birch and Swinnerton-Dyer conjecture in higher analytic rank; but see the second footnote in Chapter 1.)

See also [5.1] for further general remarks of the still embryonic "Chow-Heegner point" program.

5.2 Generalised Heegner cycles

Here we shall recall the attempt made in [BDP2] to pursue this plan in the particular case where E is an elliptic curve with complex multiplication and V is the product of a suitable number of copies of E and a Kuga-Sato variety.

For $\epsilon = 0, 1$, let $X_\epsilon(N)$ be the modular curve over $\mathbb{Q}$ introduced in Chapter 2. As shown e.g. in [5.5], it is the *coarse* (but not *fine* if $\epsilon = 0$ or $N \leq 2$) moduli space associated to the problem of classifying pairs $(A, \star)$ of (generalised) elliptic curves equipped with a cyclic subgroup of order N of the N-torsion subscheme $A[N]$ of A (resp. a point on A of exact order N) if $\epsilon = 0$ (resp. if $\epsilon = 1$). The reader may wish to consult e.g. [5.4] and [5.5] for general background on these moduli spaces. In a nutshell, if

$$\mathcal{F}_\epsilon : \{\text{Schemes over } \mathbb{Q}\} \to \{\text{Sets}\}$$

is the corresponding moduli functor, there is a natural transformation

$$\Psi : \mathcal{F}_\epsilon \longrightarrow \mathrm{Hom}(-, X_\epsilon(N)),$$

universal among such, but which may fail to be an isomorphism. In particular, the curve $X_\epsilon(N)$ is not always equipped with a universal elliptic curve over it, which would correspond to a preimage of $\mathrm{Id}_{X_\epsilon(N)}$ by Ψ. One way of circumventing this technical difficulty in a uniform way is to rigidify the moduli problem by imposing extra auxiliary level structure at an integer $M \geq 3$, with $(M, N) = 1$. More precisely, let $\tilde{X}_\epsilon(N)$ denote the modular curve associated with the congruence subgroup

$$\tilde{\Gamma}_\epsilon(N) = \{\gamma \in \Gamma_\epsilon(N), \quad \gamma \equiv \mathrm{Id} \,(\mathrm{mod}\, M)\},$$

which is normal in $\Gamma_\epsilon(N)$. It is shown e.g. in [5.4] or [5.5] that the curve $\tilde{X}_\epsilon(N)$ does represent the moduli functor $\tilde{\mathcal{F}}_\epsilon$ associated with the problem of parametrizing isomorphism classes of triples $(A, \star, \varphi)$ where $(A, \star)$ is as above and $\varphi : \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z} \to A[M]$ is a choice of an isomorphism. There thus exists a universal elliptic curve

$$\pi : \tilde{\mathcal{E}}_\epsilon(N) \longrightarrow \tilde{X}_\epsilon(N) \tag{5.7}$$

which is non-singular and proper over $\mathbb{Q}$, equipped with universal level structures. In particular, any flat family of triples $(A, \star, \varphi)$ over a base T yields an element of $\tilde{\mathcal{F}}_\epsilon(T)$ and gives rise to a unique morphism of schemes $T \longrightarrow \tilde{X}_\epsilon(N)$. The assignment

$$(A, \star, \varphi) \mapsto \tilde{\mathcal{E}}_\epsilon(N) \times_\eta T,$$

in the special case where $T = \mathrm{Spec}(K)$ is the spectrum of a field K, provides natural identifications

$$\tilde{X}_\epsilon(N)(K) = \{(A, \star, \varphi)\}/\simeq, \qquad \tilde{\mathcal{E}}_\epsilon(N)(K) = \{(A, P, \star, \varphi)\}/\simeq, \tag{5.8}$$

where A/K is a generalised elliptic curve defined over K. Here, $\star$ and φ are level structures which remain invariant under the natural action of the galois group G_K on them, and P is a K-rational point on A. There is an obvious notion of isomorphism between such tuples; the curve $\tilde{X}_\epsilon(N)$ and the surface $\tilde{\mathcal{E}}_\epsilon(N)$ classify them up to isomorphism over an algebraic closure of K.

In order to lighten the notation, when the precise values of $\star$ and φ are irrelevant for the discussion, we shall refer to a point on $\tilde{X}_\epsilon(N)$ simply as $x = [A]$, and to a point on $\tilde{\mathcal{E}}_\epsilon(N)$ as $x = [A, P]$, provided the ambiguity it causes is harmless for the discussion. Over $K = \mathbb{C}$, for instance, the rule

$$(z, \tau) \mapsto [A_{(z,\tau)} = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau, P_{(z,\tau)} = z \,(\mathrm{mod}\ \mathbb{Z} + \mathbb{Z}\tau), \star_{(z,\tau)}, \varphi_{(z,\tau)}] = [A_{(z,\tau)}, P_{(z,\tau)}]$$

yields an isomorphism of complex analytic varieties

$$\mathbb{Z}^2 \rtimes \tilde{\Gamma}_\epsilon(N) \backslash \mathbb{C} \times (\mathcal{H} \cup \mathbf{P}^1(\mathbb{Q})) \simeq \tilde{\mathcal{E}}^{\mathrm{an}}_{\epsilon,\mathbb{C}}(N), \tag{5.9}$$

where $(m = (m_1, m_2), \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \in \mathbb{Z}^2 \rtimes \tilde{\Gamma}_\epsilon(N)$ acts as

$$^{(m,\gamma)}(z, \tau) = (\frac{z}{c\tau + d} + m_1 + m_2\gamma \cdot \tau, \gamma \cdot \tau).$$

For any $r \geq 0$ let

$$\tilde{\mathcal{E}}^r_\epsilon(N) = \tilde{\mathcal{E}}_\epsilon(N) \times_{\tilde{X}_\epsilon(N)} \times \cdots \times_{\tilde{X}_\epsilon(N)} \tilde{\mathcal{E}}_\epsilon(N)$$

denote the r-fold fibered product of $\tilde{\mathcal{E}}_\epsilon(N)$ over $\tilde{X}_\epsilon(N)$. For $r = 0$ and 1 we recover $\tilde{X}_\epsilon(N)$ and $\tilde{\mathcal{E}}_\epsilon(N)$ respectively. For $r \geq 2$, the variety $\tilde{\mathcal{E}}^r_\epsilon(N)$ is singular and we may replace it by its canonical desingularization described in Lemmas 5.4, 5.5 of [5.2] and §3 of [5.6], keeping the same letter to denote it. Since both are isomorphic on a dense open subset, it is still harmless and convenient to write down its points as $x = [A, P_1, \ldots, P_r] \in \tilde{\mathcal{E}}^r_\epsilon(N)(K)$, where $\{P_i\} \subseteq A(K)$.

The level M structure on $\tilde{\mathcal{E}}^r_\epsilon(N)$ induces an action of the group $\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ by translations, which allows us to define the idempotent

$$e_1 = \sum_{(a,b) \in (\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z})^r} \frac{\sigma_{a,b}}{M^{2r}} \tag{5.10}$$

on $\tilde{\mathcal{E}}_\epsilon^r(N)$. Similarly, there is a natural action of the semi-direct product $\mu_2^r \rtimes \Sigma_r$ on $\tilde{\mathcal{E}}_\epsilon^r(N)$, where Σ_r is the group of permutations of the fibers and each copy of the cyclic group μ_2 of order 2 acts by inversion on the corresponding fiber. This again gives rise to a second idempotent

$$e_2 = \sum_{\sigma \in \mu_2^r \rtimes \Sigma_r} \frac{\mathrm{sign}(\sigma)\sigma}{2^r r!}. \tag{5.11}$$

Both idempotents commute and thus $e := e_1 e_2$ is again an idempotent on $\tilde{\mathcal{E}}_\epsilon^r(N)$. As shown in [5.6], there is an isomorphism

$$S_{r+2}(\tilde{\Gamma}_\epsilon) \simeq eF^{r+1}H_{dR}^{r+1}(\tilde{\mathcal{E}}_\epsilon^r(N)), \quad f(q) \mapsto (2\pi i)^{r+1} f(q) dz_1 \cdots dz_r dq/q. \tag{5.12}$$

One may descend to the classical modular curves $X_\epsilon(N)$ by considering the group

$$G = \tilde{\Gamma}_\epsilon / \Gamma_\epsilon(N) \simeq \mathbf{PGL}_2(\mathbb{Z}/M\mathbb{Z}),$$

which is the Galois group of the covering $\tilde{X}_\epsilon(N) \longrightarrow X_\epsilon(N)$. Hence $\tilde{X}_\epsilon(N)^G \simeq X_\epsilon(N)$ and we can define $\mathcal{E}_\epsilon(N) = \tilde{\mathcal{E}}_\epsilon(N)^G$, which is still proper over $\mathbb{Q}$ but fails to represent the functor $\mathcal{F}_\epsilon$ when $\epsilon = 0$ or $N \leq 2$, and might contain singularities. Finally, for $r \geq 1$, define

$$\mathcal{E}_\epsilon^r(N) = \mathcal{E}_\epsilon(N) \times_{X_\epsilon(N)} \times \cdots \times_{X_\epsilon(N)} \mathcal{E}_\epsilon(N)$$

and replace it by its canonical desingularization. The resulting smooth variety is called the r-th *Kuga-Sato variety* over $X_\epsilon(N)$. By the same abuse of notation as above, we shall continue to refer to label its points by (r_1)-tuples $x = [A, P_1, \ldots, P_r]$, disregarding in the notation the level structure and the fact that this system of labels is only meaningful on a dense open subset of the variety. As in (5.12), we have

$$S_{r+2}(\Gamma_\epsilon(N)) \simeq e\,\mathrm{Fil}^{r+1} H_{dR}^{r+1}(\mathcal{E}_\epsilon^r(N)) \simeq e_G e\,\mathrm{Fil}^{r+1} H_{dR}^{r+1}(\tilde{\mathcal{E}}_\epsilon^r(N)), \tag{5.13}$$

where

$$e_G := \frac{1}{\sharp G} \sum_{g \in G} g$$

is the idempotent associated to $G = \mathrm{Aut}(\tilde{X}_\epsilon(N)/X_\epsilon(N))$.

Let now E be an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K. For simplicity we assume as in [BDP2, §2.6] that the the discriminant D of K is odd, its class number is $h = 1$ and that the group of units of $\mathcal{O}_K$ is $\{\pm 1\}$. Let

$$\chi_K : (\mathbb{Z}/D\mathbb{Z})^\times \simeq (\mathcal{O}_K/\sqrt{D}\mathcal{O}_K)^\times \longrightarrow \mu_2$$

denote the quadratic Dirichlet character associated with K. Assume in addition that E is the elliptic curve of conductor $N = D^2$ over $\mathbb{Q}$ which Gross canonically attached in his Ph.D thesis [5.3] to the Hecke character

$$\nu : I_K \longrightarrow K^\times, \qquad \nu((a)) = \chi_K(a\,(\mathrm{mod}\sqrt{D}\mathcal{O}_K)) \cdot a \in K^\times \tag{5.14}$$

of type $(1,0)$.

As was known since the work of Deuring, the elliptic curve $E/\mathbb{Q}$ is modular and the modular form corresponding to E by (2.10) is $f_E = \theta_\nu$, the theta series attached to ν. In fact, for $r \geq 1$,

$$\theta_{\psi^{r+1}} \in S_{r+2}(\Gamma_r), \qquad \text{where} \qquad \Gamma_r = \begin{cases} \Gamma_0(D^2) & \text{if } r \text{ is even,} \\ \Gamma_1(D) & \text{if } r \text{ is odd.} \end{cases} \tag{5.15}$$

In other words, thanks to (5.13), the cusp form $\theta_{\psi^{r+1}}$ gives rise to a holomorphic differential $(r+1)$-form on the r-th Kuga-Sato variety $\mathcal{E}^r = \begin{cases} \mathcal{E}_0^r(D^2) & \text{if } r \text{ is even,} \\ \mathcal{E}_1^r(D) & \text{if } r \text{ is odd.} \end{cases}$

Define

$$V_r = \mathcal{E}^r \times E^r. \tag{5.16}$$

It is a non-singular proper variety of dimension $d = 2r+1$ over $\mathbb{Q}$. If Tate's conjectures [5.7] hold true for $V_r \times E$, it can be shown (cf. [BDP2, §3.4]) that there exists a cycle

$$\Pi_r^? \in \mathrm{CH}^{r+1}(V_r \times E)(K)$$

such that the pair $(V_r, \Pi_r^?)$ is non-trivial in the sense of (5.6). In particular it induces a commutative diagram as in (5.4) with $c = r+1$. In addition, the variety V_r comes equipped with a natural supply of *generalised Heegner cycles* Δ_φ, as they are called in [BDP2]. These cycles are essentially in bijection with isogenies $\varphi : A \longrightarrow E$, where A ranges over elliptic curves with complex multiplication by some order in K. Specifically, let

$$\mathrm{Graph}(\varphi)^r \subset (A \times E)^r = A^r \times E^r \subset V_r$$

be the r-fold product of the graph of φ. Note that here we make use of our convention of denoting points on $\mathcal{E}^r$ simply as $x = [A, P_1, \ldots, P_r]$. In line with this convention, by $A^r \subset \mathcal{E}^r$ we mean the whole fibre of $\pi : \mathcal{E}^r \longrightarrow X$ above the point $[A]$ in $X_0(D^2)$ if r is even (resp. in $X_1(D)$ if r is odd). Letting H_φ/K denote the class field of K over which both A and φ are defined, the cycle $\Delta_\varphi := e\mathrm{Graph}(\varphi)^r$ is a null-homologous cycle of codimension $r+1$ on V_r, which is rational over H_φ. This allows us to define the *Chow-Heegner point*

$$P_{r,\varphi}^? := \pi^?(\Delta_\varphi) \overset{?}{\in} E(H_\varphi). \tag{5.17}$$

Although the existence of such a global point is conditional on the existence of $\Pi^?$, the right hand side of (5.4) is available unconditionally thanks to (5.15), and the point

$$P_{r,\varphi,\mathbb{C}} = \mathrm{AJ}_{\mathbb{C}}(P_{r,\varphi}^?) \in \mathbb{C}/\Lambda_E$$

can be computed numerically. For instance, if we take $A = E$ and $\varphi = \mathrm{Id}_E$, then

$$P_{r,\mathrm{Id},\mathbb{C}} = \Omega^{-r} \frac{(2\pi i)^{r+1}}{(\tau - \bar{\tau})^r} \int_{i\infty}^{\tau} (z - \bar{\tau})^r \theta_\nu(z) dz \in \mathbb{C}/\Lambda_E$$

where $E_{\mathbb{C}}^{\mathrm{an}} = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ with $\tau = \Omega_1/\Omega \in \mathcal{H}\cap K$ for suitable periods Ω_1, Ω of E. See [BDP2, §5] for more details, including a summary of some numerical calculations of the points $P_{r,\varphi,\mathbb{C}}$ that are carried out for different choices of E and r.

Remark 5.1. The goal of the Student project at the Arizona Winter School is precisely to carry out the same sorts of calculations for cycles arising in in the product of three Kuga-Sato varieties. These *triple product cycles* are introduced and discussed in the next chapter.

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Chapter 6

Motives and L-functions

It will be useful to place the study of Chow-Heegner points in a somewhat broader setting concerned with special values of L-functions attached to motives. This chapter contains a brief overview of these concepts, although the notion of motives will continue to be approached in a rather loose and informal way throughout these notes.

6.1 Motives

A *motive over a field* F is essentially a piece of the cohomology of a variety over F. Motives grew out of Weil's fundamental insight that an algebraic variety V over a field F gives rise to a plethora of cohomology groups (satisfying the axioms of what are now commonly referred to as *Weil cohomology theories*). There are, for instance, the Betti cohomology $H^*_B(V(\mathbb{C}), \mathbb{Q})$ of the underlying complex variety (relative to a complex embedding of F), the algebraic De Rham cohomology $H^*_{dR}(V/F)$ of V over F described in Section 4.2, and the étale cohomology groups $H^*_{\mathrm{et}}(V_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ for each rational prime ℓ. These objects are of quite different nature—they are finite-dimensional vector spaces over $\mathbb{Q}$, F, and $\mathbb{Q}_\ell$ respectively, equipped eventually with various extra structures, such as the Hodge filtration on $H^*_{\mathrm{dR}}(V/F)$ described in Section 4.2 and the continous action of G_F on the ℓ-adic étale cohomology. Nevertheless these cohomology groups are related through various comparison isomorphisms, and therefore display certain common features. One would like to view them as the diverse cohomological avatars of a common mathematical object. This (as yet to be defined) object is the *motive* attached to the variety V.

Somewhat more generally, elements of the ring $\mathcal{C}(V, V)$ of algebraic correspondences from V to itself induce linear transformations on the cohomology of V, by functoriality. Fixing an idempotent e in $\mathcal{C}(V, V)$, one would likewise want to view the vector spaces

$$eH^*_B(V(C)), \qquad eH^*_{\mathrm{dR}}(V/F), \qquad eH^*_{\mathrm{et}}(V_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) \tag{6.1}$$

as the common manifestations of an underlying motive. We are led to the following precise mathematical definition.

Definition 6.1. The category $\mathcal{M}_{F,\mathbb{Q}}$ of *Chow motives* over F with coefficients in $\mathbb{Q}$ is the category whose objects are triples (V, e, m), where

- V is a smooth projective variety of dimension d;
- $e \in \mathrm{CH}^d(V, V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an idempotent in the ring of algebraic correspondences from V to itself;
- m is an integer.

The morphisms between two objects are given by

$$\mathrm{Hom}_{\mathcal{M}_{F,\mathbb{Q}}}((V, e, m), (V', e', m')) = e' \cdot \mathrm{CH}^{d+m'-m}(V \times V') \otimes \mathbb{Q} \cdot e.$$

The vector spaces in (6.1) are called the *Betti, de Rham, and étale realisations* of the motive $M = (V, e, 0)$, which can be thought of as the *e-isotypic part of the cohomology of* V. For an explanation of the role played by the integer m in the definition of Chow motive, see the discussion below on Tate motives and twists.

It is often useful to allow e to be an idempotent in $\mathcal{C}(V, V) \otimes_{\mathbb{Q}} E$, for some field E. In that case, one speaks of a "motive over F with coefficients in E", and denotes by $\mathcal{M}_{F,E}$ for the category of such motives.

The category of motives is an abelian tensor category, equipped with all the familiar operations of linear algebra (passage to the dual, tensor products and internal homs). For instance, the dual of a motive M is the motive $M^\vee$ whose various realisations are the linear duals of the corresponding realisations of M.

From now on, let us assume that the base field F is the rational field $\mathbb{Q}$ since only motives defined over this base field will intervene in these notes. (Of course, the reader should have no difficulty in transposing the remainder of this section to the case of a more general base field.)

Let $M = (V, e, 0)$ be a motive arising from a variety V defined over $\mathbb{Q}$. Letting S denote the product of the prime numbers at which V has bad reduction, it is known that

1. the ℓ-adic étale realisations M_ℓ are unramified at all primes $p \nmid \ell S$;
2. for such p, the characteristic polynomial of the geometric frobenius element Frob_p acting on M_ℓ has rational coefficients, and *does not depend on* ℓ. Its roots $\alpha_{p,1}, \ldots \alpha_{p,d}$ can therefore be viewed as complex numbers. The polynomials

   $$L_{M,p}(T) = (1 - \alpha_{p,1}T) \cdots (1 - \alpha_{p,d}T) \in \mathbb{Q}[T]$$

   indexed by the rational primes $p \nmid S$ are among the most fundamental invariants attached to M. One can also define polynomials $L_{M,p}(T)$ for $p|S$ by considering the action of the frobenius element at p on the invariants of M_ℓ (for $\ell \neq p$) under the inertia group at p. It is also true (although this fact lies somewhat deeper) that the polynomials $L_{M,p}(T)$ attached to the bad primes, which are of degree $< d$, also have rational coefficients and are independent of ℓ.

The above properties are summarised by saying that the collection of M_ℓ as ℓ varies is a *compatible system of ℓ-adic representations*. When M arises in the j-th cohomology of a smooth projective variety, it is known that the roots $\alpha_{p,i}$ $(1 \le i \le d)$ are complex numbers of absolute value $p^{j/2}$. In that case, one says that M is *pure of weight j*.

The *Hodge-Tate weights* of a motive M are defined by considering the p-adic valuations of the α_i rather than their complex absolute values. These Hodge-Tate weights are simply the collection of rational numbers $\mathrm{ord}_p(\alpha_{p,i})$, counted with multiplicities.

We refer the reader to [6.1], [6.4], [6.5], or [6.6], for further background on the general theory of motives. For the reader of these notes, it is perhaps more important to keep in mind the following important classical examples of motives, which are also the most studied and the best understood:

1. Dirichlet and Artin motives. These arise from the variety $V = \mathrm{Spec}(F)$, where F is a finite (Galois) extension of $\mathbb{Q}$, and $\mathrm{Spec}(F)$ is viewed as a variety over $\mathbb{Q}$, on which elements of $G = \mathrm{Gal}(F/\mathbb{Q})$ acts as automorphisms. Let

$$\rho : \mathrm{Gal}(F/\mathbb{Q}) \longrightarrow \mathbf{GL}_d(E),$$

be an irreducible representation of $\mathrm{Gal}(F/\mathbb{Q})$, and let e_ρ be the corresponding idempotent in the group ring $E[G]$, viewed is contained in $\mathcal{C}(V, V) \otimes_{\mathbb{Q}} E$. The triple $(V, e_\rho, 0)$ is called the *Artin motive* attached to ρ and is denoted M_ρ. It is a motive defined over $\mathbb{Q}$ with coefficients in E. Its ℓ-adic realisations are free $E \otimes \mathbb{Q}_\ell$-modules of rank $d = \dim(\rho)$, obtained by tensoring the E-vector space V_ρ underlying the finite-dimensional Galois representation ρ with $\mathbb{Q}_\ell$. It is pure of weight 0, as can be seen directly, since the $\alpha_{p,i}$ are roots of unity. (Note also that V is a zero-dimensional variety, and its cohomology is therefore concentrated in degree 0.) Likewise the Hodge-Tate weights of M_ρ are $(0, \ldots, 0)$ (d times). When ρ is one-dimensional (and therefore correponds to a Dirichlet character χ) one also calls M_ρ the *Dirichlet motive* attached to χ.

2. Tate motives. The motive $\mathbb{Z}(-1)$ is the rank one motive corresponding to the triple $(\mathbf{P}^1_{\mathbb{Q}}, e, 0)$ where e is an idempotent which annihilates $H^0(\mathbf{P}^1_{\mathbb{Q}})$ and acts as 1 on $H^2(\mathbf{P}^1_{\mathbb{Q}})$ for any Weil cohomology theory. Recall that $H^1(\mathbf{P}^1_{\mathbb{Q}}) = 0$ because $\mathbf{P}^1_{\mathbb{Q}}$ has genus 0.

The motive $\mathbb{Z}(1)$ is then defined as the *dual* of $\mathbb{Z}(-1)$. It is pure of weight -2 and its (single) Hodge-Tate weight is -1.

Its ℓ-adic realisation is isomorphic to the ℓ-adic Tate module of the multiplicative group $\mathbf{G}_m$, on which the geometric frobenius element at p acts as multiplication by p^{-1}.

The deRham realisation of $\mathbb{Z}(-1)$ is the one-dimensional $\mathbb{Q}$-vector space $H^2_{\mathrm{dR}}(\mathbf{P}^1_F)$. In terms of hypercocycles as introduced in (4.6) (and worked out more explicitly for the first cohomology group of a curve in (4.10)), it is spanned by the class of the hypercocycle

$$(0, 0, dt/t, 0) \in \Omega^2_U \times \Omega^2_V \times \Omega^1_{U\cap V} \times 0, \qquad U = \mathbf{P}^1 - \{0, \}, \quad V = \mathbf{P}^1 - \{\infty\}.$$

The motive $\mathbb{Z}(j)$ is the j-th tensor power of $\mathbb{Z}(1)$ with itself. It is pure of weight $-2j$ and its Hodge-Tate weight is $-j$.

3. Tensor products. One may construct new motives from existing ones by taking duals and tensor products. If M_1 and M_2 are pure motives of weights j_1 and j_2, then the motives $M_1 \otimes M_2$ and the dual motive $M_1^\vee$ are pure of weight $j_1 + j_2$ and $-j_1$ respectively. The most important basic instance of the tensor product construction is the j-th Tate twist of a motive M, defined as

$$M(j) := M \otimes \mathbb{Z}(j) := (V, e, j) \qquad \text{if } M = (V, e, 0).$$

(Thus, the purpose of the third entry in the description of an object in the category of motives is to "keep track of Tate twists".) In particular, if M is pure of weight k then the Tate twist $M(j)$ is pure of weight $k - 2j$. Another important example is the *Kummer dual*

$$M^* := \hom(M, \mathbb{Z}(1)) = M^\vee(1),$$

which is pure of weight $-2-k$. When M is isomorphic to its Kummer dual M^*, we say that M is *polarised* or *self-dual.* Note that a self-dual motive is necessarily pure of weight -1.

4. Elliptic curves. If E is an elliptic curve over $\mathbb{Q}$, and

$$e = \frac{1 - [-1]}{2}$$

is the idempotent in which $[-1]$ denotes multiplication by -1 on E, then the triple

$$h^1(E) := (E, e, 0)$$

is called the *motive attached to* E. It is a motive of rank two with coefficients in $\mathbb{Q}$, which is pure of weight 1. Its Hodge-Tate weights are $(0, 1)$ if $E \pmod p$ is an ordinary elliptic curve, and $(1/2, 1/2)$ if E is supersingular at p.

If E is given by a non-singular Weierstrass equation of the form

$$y^2 = 4x^3 + ax + b,$$

then the de Rham realisation of M has a $\mathbb{Q}$ basis represented by the classes of the hypercocycles

$$\begin{aligned} \omega &= \left(\frac{dx}{y}, \frac{dx}{y}, 0\right) \in \Omega^1_U \times \Omega^1_V \times \mathcal{O}_{U\cap V}, \\ \eta &= \left(\frac{xdx}{y}, \frac{xdx}{y} - d\frac{y}{2x}, \frac{y}{2x}\right) \end{aligned}$$

attached to the affine cover $U = E - \{0\}, V = E - E[2]^*$, where $E[2]^*$ denoted the set of points of E of (exact) order 2. The Hodge filtration of $H^1_{\mathrm{dR}}(E/\mathbb{Q})$ is described by

$$\mathrm{Fil}^j\, H^1_{\mathrm{dR}}(E/\mathbb{Q}) = \begin{cases} H^1_{\mathrm{dR}}(E/\mathbb{Q}) & \text{if } j \le 0; \\ \mathbb{Q}\omega & \text{if } j = 1; \\ 0 & \text{if } j \ge 2. \end{cases}$$

Finally, the Poincaré duality

$$h^1(E) \times h^1(E) \longrightarrow h^2(\mathbf{P}^1) = \mathbb{Z}(-1),$$

shows that the the motive

$$h^1(E)(1) := (E, e, 1) = h^1(E)^\vee$$

of weight -1 is self-dual. It is the prototypical example of a self-dual motive.

5. Scholl motives of Modular forms. Let $f \in S_k(N, \chi)$ be a normalised eigenform of weight $k = r + 2$, level N and character χ with coefficients in a number field K_f. Scholl has attached to f a motive $M_f \in \mathcal{M}_{\mathbb{Q},K_f}$ over $\mathbb{Q}$ with coefficients in K_f. This motive was for the most part already described in Chapter 5, and is of the form $M_f = (V, e, 0)$, where

- V is the r-th Kuga-Sato variety $\mathcal{E}_1^r(N)$ over the modular curve $X_1(N)$;
- e is a suitable projector constructed from Hecke operators (whose graph might be regarded as a cycle of codimension $k-1$ on $V \times V$) and automoprhisms of V, giving rise to the idempotent which projects onto the f-isotypical component $S_k(N, \chi)[f]$ of $S_k(N, \chi)$.

The ℓ-adic étale realisation of M_f is just Deligne's two-dimensional ℓ-adic representation attached to f, which occurs in the f-isotypic component of $H^{r+1}_{\text{et}}(V_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ for the action of the Hecke correspondences. In particular, this ℓ-adic realisation is a free module of rank two over $K_f \otimes \mathbb{Q}_\ell$. The Frobenius eigenvalues $\alpha_{p,1}$ and $\alpha_{p,2}$ are the roots of the polynomial $x^2 - a_p(f)x + \chi(p)p^{k-1}$, where $a_p(f)$ is the p-th fourier coefficient of f, and therefore

$$L_{M_f,p}(T) = 1 - a_p(f)T + \chi(p)p^{k-1}T^2.$$

The motive M_f is pure of weight $k-1$ and its Hodge-Tate weights are $(0, k-1)$ when $a_p(f)$ is a p-adic unit.

It follows from (5.13) that the de Rham realization of M_f is

$$H^*_{\text{dR}}(M_f) = eH^{r+1}_{\text{dR}}(\mathcal{E}_1^r(N))[f],$$

and its Hodge filtration is given by

$$\mathrm{Fil}^j\, H^*_{\text{dR}}(M_f) = \begin{cases} H^*_{\text{dR}}(M_f) & \text{if } j \le 0; \\ S_k(\Gamma_1(N))[f] = K_f \cdot f & \text{if } 1 \le j \le k-1; \\ 0 & \text{if } j \ge k. \end{cases}$$

We draw the novice reader's attention to a potentially confusing clash in terminologies: the weight of a modular form f is 1 greater than the weight of its associated motive M_f.

When k is even, the arithmetic of the motive $M_f(k/2)$ of weight -1 is particularly interesting. This motive is self-dual when f has trivial nebentypus character. In that case, the motive $M_f(k/2)$ is a natural generalisation of the self-dual motive $h^1(E)(1)$ attached to an elliptic curve, which one recovers when f is a eigenform of weight 2 with rational fourier coefficients.

6. Tensor products of Scholl motives. If $f_1, \ldots, f_n$ are n cusp forms of weights $k_1, \ldots, k_n$, one can consider the tensor product

$$M(f_1, \ldots, f_n) := M_{f_1} \otimes M_{f_2} \otimes \cdots \otimes M_{f_n}$$

of the Scholl motives attached to these modular forms. It is a motive of rank 2^n and is pure of weight $k_1 + \cdots + k_n - n$. The cases where $n = 2$ or $n = 3$ turn out to be particularly interesting for us, and indeed one of the main themes of the next chapter is the study the motive $M(f_1, f_2, f_3)$. More precisely, we will be specially interested in the motive $M(f_1, f_2, f_3)(\frac{k_1+k_2+k_3-2}{2})$ when $k_1 + k_2 + k_3$ is even, which is pure of weight -1 and is furthermore self-dual when $\chi_1\chi_2\chi_3 = 1$.

6.2 L-functions

If M is a motive over $\mathbb{Q}$ with coefficients in a field $E \subset \mathbb{C}$, we may attach to it the complex L-function

$$L(M, s) = \prod_p P_{M,p}(p^{-s})^{-1}, \tag{6.2}$$

defined by an Euler product over all the rational primes. This Euler product converges absolutely on the right half-plane $\mathrm{Re}(s) > 1 + w/2$ when M is pure of weight w. It is conjectured that $L(M, s)$ extends to a meromorphic function on the entire complex plane, with a functional equation relating the values $L(M, s)$ and $L(M^*, -s)$. This conjecture (which is quite deep in general, but is known for all the motives listed as examples in the previous section) can be used to define the values of $L(M, s)$ in full generality (with the usual convention that $L(M, j) = \infty$ if $L(M, s)$ has a pole at $s = j$). In particular, we will set $L(M) := L(M, 0)$ and call this the *L-value attached to the motive M*.

A motive is said to be *in the range of classical convergence* if the Euler product defining $L(M)$ converges absolutely, so that analytic continuation of $L(M, s)$ is not required to define it. For such motives, the value $L(M)$ is of course always finite and non-zero.

If the Γ-factors in the functional equation relating $L(M)$ to $L(M^*)$ have neither zero nor poles (so that in particular the order of vanishing of $L(M, s)$ and $L(M^*, -s)$ at $s = 0$ are the same) then the motive M is said to be *critical* in the sense of Deligne. The fundamental conjectures of Deligne predict that for critical motives, the special value $L(M)$ can always be expressed as the product of a specific period Ω_M arising in the comparison isomorphism between the Betti realisation M_{B} and the deRham realisation M_{dR} of M. For a precise description of Ω_M, see [6.3].

Here are a few basic examples illustrating these concepts.

1. The Riemann zeta function. The L-function of the trivial motive $Z(0)$ is the classical Riemann zeta-function $\zeta(s)$. In particular, the motive $\mathbb{Z}(j)$ is critical if and only if j is even and ≥ 2 or odd and strictly negative. For such critical $j \geq 0$, the period attached to M is $(2\pi i)^j$, as was first proved by Euler.

2. Dirichlet and Artin L-series. More generally, one recovers the classical Dirichlet L-series $L(\chi, s)$ and the Artin L-series $L(\rho, s)$ by taking the L-series attached to the Dirichlet motive $Z(\chi)$ and the Artin motive M_ρ respectively.

3. Hasse-Weil L-series of elliptic curves. The L-series $L(h^1(E), s)$ attached to the motive $h^1(E)$ is the Hasse-Weil L-series $L(E, s)$ attached to E. The functional equation for $L(E, s)$ (which follows from the modularity of E) shows that $h^1(E)$ has only one critical Tate twist: the motive $h^1(E)(1)$ whose ℓ-adic realisation is the ℓ-adic Tate module of E (tensored with $\mathbb{Q}_\ell$). The integral representation of $L(E, s) = L(f, s)$, combined with the corresponding modular parametrisation $X_0(N) \longrightarrow E$, can be used to show that

$$L(h^1(E)(1)) = \mathscr{L} \cdot \Omega_{h^1(E)(1)}, \tag{6.3}$$

where

- $\mathscr{L}$ is a (possibly zero!) rational number;
- the period $\Omega_{h^1(E)(1)}$ is given by

  $$\Omega_{h^1(E)(1)} = \int_{E(\mathbb{R})} \omega_E,$$

  in which
- ω_E is a non-zero regular differential over $\mathbb{Q}$, i.e., a generator of $\mathrm{Fil}^1\, h^1(E)(1)_{\mathrm{dR}}$;
- the region $E(\mathbb{R})$ of integration generates the subspace of $h^1(E)(1)_{\mathrm{B}}$ invariant under the action of complex conjugation.

It is important to remark that one would be quite at a loss to prove (6.3) if one did not know that the elliptic curve E is *modular*. This is, in fact, a general pattern. As will be explained more later, one knows *virtually nothing* about the complex numbers $L(M)$ (aside from the fact that they are defined and non-zero, when M lies in the range of complex interpolation) without first harnessing the analytic properties of $L(M, s)$ by relating M to automorphic forms.

4. Hecke L-series of modular forms. The Hecke L-series $L(f, s)$ attached to a modular form $f \in S_k(\Gamma_1(N), \chi)$ is equal to the L-series $L(M_f, s)$ of the associated Deligne-Scholl motive M_f. It is given by the Euler product

$$L(f, s) = \prod_p L^{(p)}(f, p^{-s})^{-1}, \quad \text{with } L^{(p)}(f, T) = (1 - \alpha_1(p)T)(1 - \alpha_2(p)T), \tag{6.4}$$

where the complex parameters $\alpha_1(p)$ and $\alpha_2(p)$ are determined by

$$\alpha_1(p) + \alpha_2(p) = a_f(p), \qquad \alpha_1(p)\alpha_2(p) = \chi(p)p^{k-1}. \tag{6.5}$$

Hecke's functional equation for $L(f,s)$ implies that the motive $M_f(j)$ is critical precisely when $1 \leq j \leq k-1$, and is self-dual when $k = 2j$ and the nebentype character χ is trivial.

5. Rankin L-series. If f and g are modular forms of weight k and ℓ with $k > \ell$, the L-function $L(M_f \times M_g, s)$ is called the Rankin L-series attached to f and g. The motive $M_f \otimes M_g(j)$ is critical if and only if the integer j lies in the closed interval $[\ell, k-1]$. The Deligne periods for these critical Tate twists are essentially given by the Petersson scalar product of the form f of *higher weight*:

$$\langle f, f \rangle := \int_{\mathcal{H}/\Gamma_0(N)} f(z)\overline{f(z)} y^{k-2} dz d\bar{z},$$

up to multiplication by powers of $(2\pi i)$ and elements of $K_g K_f$. Deligne's conjectures about the "transcendental part" of $L(M_f \otimes M_g(j))$ for $\ell \leq j \leq k-1$ can be shown using the integral representation for $L(M_f \otimes M_g, s)$ that arises from Rankin's method.

5. Triple product L-functions. Let

$$f_i = \sum_{n \geq 1} a_i(n) q^n \in S_{k_i}(N_i, \chi_i), \quad i = 1, 2, 3,$$

be three normalized primitive cuspidal eigenforms of levels $N_i \geq 1$, weights $k_1 \geq k_2 \geq k_3 \geq 1$, and Nebentypus characters χ_i. For $j = 1, 2, 3$, let $\alpha_1(f_j, p)$ and $\alpha_2(f_j, p)$ denote the complex parameters attached to the form f_j and the prime p that arise in (6.4) and (6.5). The *incomplete L-function* of the motive $M(f_1, f_2, f_3)$, denoted by $L(f_1, f_2, f_3; s)$, is given by the incomplete Euler product

$$L(f_1, f_2, f_3; s) = \prod_{p \nmid N} L^{(p)}(f_1, f_2, f_3; p^{-s})^{-1},$$

where

$$L^{(p)}(f_1, f_2, f_3; T) = \prod_{\tau} (1 - \alpha_{\tau(1)}(f_1, p) \cdot \alpha_{\tau(2)}(f_2, p) \cdot \alpha_{\tau(3)}(f_3, p) T),$$

and τ runs over the eight possible maps from the set $\{1, 2, 3\}$ to the set $\{1, 2\}$. In 1987, Garrett [7.3] and Piatetski-Shapiro and Rallis [7.9] gave an integral representation for the triple product L-series $L(f_1, f_2, f_3; s)$ which allowed them to show that it admits a meromorphic continuation to the whole complex plane.

Throughout these notes, we shall make the following hypothesis, which the reader may regard as an analogue of the *Gross-Zagier hypothesis* (3.8) considered in Chapter 3.

Definition 6.2. The triple (f_1, f_2, f_3) of modular forms is said to satisfy the *Gross-Prasad hypothesis* if

(i) $\chi_1 \cdot \chi_2 \cdot \chi_3 = 1$, so that in particular $k_1 + k_2 + k_3$ is even.

(ii) There exists no prime p dividing $gcd(N_1, N_2, N_3)$ at which

(a) the local automorphic representations π_{f_i} of $\mathbf{GL}_2(\mathbb{Q}_p)$ associated to f_1, f_2 and f_3 lie each in the discrete series, and

(b) there exists no $\mathbf{GL}_2(\mathbb{Q}_p)$-invariant linear form in the tensor product $\pi_{f_1} \otimes \pi_{f_2} \otimes \pi_{f_3}$.

More general versions of this hypothesis, in the vein of the Heegner hypothesis introduced in Definition 3.1, could be considered; in the current, still preliminary version of these notes, we preferred to stay in this somewhat simpler setting in order to make the description of the geometrical constructions given in Chapter 7 simpler. The reader can consult [DRS] for more details on a more general scenario where the hypothesis above is relaxed.

We refer the reader to [7.1], [7.2], [7.5] and [7.4] for the basic theory of automorphic forms and representations in connection with the classical theory of modular forms. Let us just make here some remarks about the terminology used in these and other treatments, which might be useful to the reader.

- The term *discrete series* is motivated by the fact that these are the representations which occur discretely in the Hilbert decomposition of the Hilbert space $\mathrm{L}_2(\mathbf{PGL}_2(\mathbb{Q}_p))$ of square-integrable functions under right and left-translation by $\mathbf{GL}_2(\mathbb{Q}_p)$. These are also the class of admissible irreducible infinite-dimensional automorphic representations of $\mathbf{GL}_2(\mathbb{Q}_p)$ which admit a Jacquet-Langlands lift to $D_p^\times$, where D_p denotes the unique division quaternion algebra over $\mathbb{Q}_p$ up to isomorphism.

- For an irreducible admissible representation π of $\mathbf{GL}_2(\mathbb{Q}_p)$, it is equivalent to say that it lies in the *discrete series*, to say it is *square-integrable* and to say that it is either *Steinberg* (also called *special*) or *supercuspidal*. Yet put another way, one says that π lies in the discrete series if and only if it does not lie in the *principal series*.

- This classification of automorphic forms often admits a pleasant interpretation in other contexts. For instance, if $\pi = \pi_f$ is the local automorphic representation associated with a eigenform $f \in S_k(N, \chi)$ as above, it turns out that π_f lies in the principal series if and only if the motive $\mathcal{M}(f)$ acquires good reduction over an abelian extension of $\mathbb{Q}_p$.

Let us assume the Gross-Prasad hypothesis. Then, the work of Piatetski-Shapiro and Rallis (cf. Theorem 5.2 of [7.9] together with D. Prasad's Ph.D thesis [7.10] show that there exist local Euler factors $L^{(p)}(f_1, f_2, f_3; s)$ (of degree < 8) at the bad primes $p \mid N$ and an archimedean factor $L^{(\infty)}(f_1, f_2, f_3; s)$ for which the completed L-function

$$\Lambda(f_1, f_2, f_3; s) := L(f_1, f_2, f_3; s) \cdot \prod_{p|N} L^{(p)}(f_1, f_2, f_3; s) \cdot L^{(\infty)}(f_1, f_2, f_3; s)$$

satisfies the functional equation

$$\Lambda(f_1, f_2, f_3; s) = \varepsilon \Lambda(f_1, f_2, f_3; k_1 + k_2 + k_3 - 2 - s) \tag{6.6}$$

where

$$\varepsilon = \begin{cases} -1 & \text{if } k_1 < k_2 + k_3 \\ +1 & \text{if } k_1 \geq k_2 + k_3. \end{cases}$$

The center of symmetry of (6.6) is thus $s_0 = \frac{k_1+k_2+k_3-2}{2}$, at which it is known there is no pole. This is consistent with the fact that the pure motive of weight -1

$$M_0(f_1, f_2, f_3) := (M_{f_1} \otimes M_{f_2} \otimes M_{f_3})(\frac{k_1 + k_2 + k_3 - 2}{2}) \tag{6.7}$$

is self-dual.

In the literature, the case where $k_1 < k_2 + k_3$ is sometimes called *balanced*, while the case where $k_1 \geq k_2 + k_3$ is called *unbalanced*. The nature of the Deligne period $\Omega_{M_0(f_1,f_2,f_3)}$ depends on whether one is in the balanced or unbalanced case. More precisely:

1. In the *balanced case* where $k_1 < k_2 + k_3$, the Deligne period of $M_0(f_1, f_2, f_3)$ is (up to powers of π) equal to
$$\langle f_1, f_1\rangle^2 \langle f_2, f_2\rangle^2 \langle f_3, f_3\rangle^2.$$

2. In the *unbalanced case* where $k_1 \geq k_2 + k_3$, the Deligne period of $M_0(f_1, f_2, f_3)$ is equal to $\langle f_1, f_1\rangle^2$.

The Deligne conjecture for these critical self-dual motives can be proved using Garrett's integral representation for the triple product L-series. See the lectures of Kartik Prasanna at this winter school where these periods are studied in much greater depth.

6.3 The Birch and Swinnerton-Dyer conjecture, revisited

The Birch and Swinnerton-Dyer conjecture discussed in Chapter 1 concerns the value $L(M)$ of the self-dual motive $M = h^1(E)(1)$ attached to the elliptic curve E. More precisely, it relates the order of vanishing of $L(h^1(E)(1), s)$ at $s = 0$ (or equivalently, the order of vanishing of $L(E, s)$ at $s = 1$) to the rank of the Mordell-Weil group $E(\mathbb{Q})$.

There is a natural generalisation of the Birch and Swinnerton-Dyer conjecture, due to Beilinson-Bloch (cf. [6.7]) that applies to an arbitrary (self-dual) motive M. In order to formulate it, it is necessary to give a "motivic" interpretation of the Mordell-Weil group of E. This interpretation is based on the remark that a homologous algebraic cycle D supported on a codimension one subvariety $Z = \{P_1, \ldots, P_r\}$ of E gives rise to a (mixed) motive $M_{E,D}$ which fits into the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^1(E) \arrow[r] \arrow[d] & M_{E,D} \arrow[r] \arrow[d] & \mathbb{Z}(-1) \arrow[r] \arrow[d, "\mathrm{cl}_D"] \arrow[dr, "0"] & 0 & \\ 0 \arrow[r] & H^1(E) \arrow[r] & H^1(E-Z) \arrow[r] & \oplus_{i=1}^r \mathbb{Z}(-1) \arrow[r] & \mathbb{Z}(-1) \arrow[r] & 0 \end{tikzcd} \tag{6.8}$$

in which the bottom row is the usual cohomology sequence relating the cohomologies of E, $E - Z$, and Z, the rightmost vertical arrow is the one which sends 1 to the class of D in Z, and the extension $M_{E,D}$ is obtained by push-out. The isomorphism class of this extension depends only on the image of D in $E(\mathbb{Q})$, and hence there is a natural map, which shall be referred to as the *motivic Abel-Jacobi map*:

$$\mathrm{AJ}_{\mathcal{M}} : \mathrm{CH}^1(E)_0(\mathbb{Q}) \longrightarrow \mathrm{Ext}^1_{\mathcal{M}_{\mathbb{Q},\mathbb{Q}}}(\mathbb{Z}(-1), h^1(E)) = \mathrm{Ext}^1_{\mathcal{M}_{\mathbb{Q},\mathbb{Q}}}(\mathbb{Z}, h^1(E)(1)),$$

where the extensions are taken in the category $\mathcal{M}_{\mathbb{Q},\mathbb{Q}}$ of motives over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$. This terminology is justified by the fact that one recovers the image $\mathrm{AJ}_\infty(D)$ of D under the usual complex Abel-Jacobi map of Section 4.3 by applying to $\mathrm{AJ}_{\mathcal{M}}(M)$ the realisation functor

$$DR_{\mathbb{C}} : \mathcal{M}_{\mathbb{Q},\mathbb{Q}} \longrightarrow \mathcal{MHS}$$

from the category of motives to the category of *integral mixed Hodge structures*. The group $\mathrm{Ext}^1_{\mathcal{MHS}}(\mathbb{Z}, h^1_{\mathrm{dR}}(E/\mathbb{C})(1))$ is identified with $E(\mathbb{C})$ in a natural way. (For details, see for example [6.2] or the brief exposition in Sections 3.5 and 3.6 of [BDP1].)

More generally, if $M = (V, 1, j) = H^{2j-1}(V)(j)$ is a pure motive of weight -1 arising in the $(2j-1)$-st cohomology of a smooth projective variety V, then there is a natural map

$$\mathrm{AJ}_{\mathcal{M}} : \mathrm{CH}^j(V)_0(\mathbb{Q}) \longrightarrow \mathrm{Ext}^1_{\mathcal{M}_{\mathbb{Q},\mathbb{Q}}}(\mathbb{Z}(-j), eH^{2j-1}(E)) = \mathrm{Ext}^1_{\mathcal{M}_{\mathbb{Q},\mathbb{Q}}}(\mathbb{Z}, M),$$

which arises by sending the cycle D supported on a codimension j subvariety Z to the class of the extension $M_{V,D}$ generalising the one in (6.8):

$$\begin{tikzcd}
0 \arrow[r] & H^{2j-1}(V)(j) \arrow[r] \arrow[d] & M_{V,D} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{cl}_D"] \arrow[dr, "0"] & 0 & \\
0 \arrow[r] & H^{2j-1}(V)(j) \arrow[r] & H^{2j-1}(V-Z)(j) \arrow[r] & H^{2j}(Z)(j) \arrow[r] & H^{2j}(V)(j) \arrow[r] & 0
\end{tikzcd}$$

One is led to simply *define* the Mordell-Weil group of M by setting

$$\mathrm{MW}(M) := e\,\mathrm{CH}^j(V)_0(\mathbb{Q}) \stackrel{?}{=} \mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Z}, M).$$

The Beilinson-Bloch conjecture (in a weak form) can be stated as follows.

Conjecture 6.3. *Let M be a self-dual motive over $\mathbb{Q}$ with coefficients in E. Then*

$$\mathrm{ord}_{s=0} L(M, s) = \dim_E \mathrm{MW}(M).$$

The following are the most basic examples of the Beilinson-Bloch conjectures.

1. Elliptic curves and (modular) abelian varieties. The motive M_f attached to a modular form f of weight two is essentially equivalent to the motive of the abelian variety $A_f/\mathbb{Q}$ (taken up to isogeny) which Eichler and Shimura attach to f as a quotient of the jacobian of $X_1(N)$. This motive was thus already considered implicitly in Chapter 3, and the Beilinson-Block conjecture for M_f is just a restatement of the Birch and Swinnerton-Dyer conjecture for the Abelian variety A_f.

2. Modular forms of higher even weight. If f is a modular form of *even* weight $k = r+2$, the motive $M_f(k/2) = (\mathcal{E}^r, e, k/2)$ attached to f is self-dual when the nebentypus character of f is trivial. In that case, the Beilinson-Bloch conjectures predict that

$$\mathrm{ord}_{s=0} L(M_f(k/2), s) = \mathrm{ord}_{s=k/2} L(f,s) = \dim_{K_f} e\,\mathrm{CH}^{k/2}(\mathcal{E}^r)_0(\mathbb{Q}).$$

When $L(f, k/2) = 0$, a non-trivial cycle can sometimes be produced in the Chow group of the Kuga-Sato variety $\mathcal{E}^r$ by considering certain *Heegner cycles* supported on the fibers in $\mathcal{E}^r$ above CM points. See [6.8] for a nice exposition of the general set-up.

3. Double Rankin products. If $f \in S_k(\Gamma_0(N), \chi_f)$ and $g \in S_\ell(\Gamma_0(M), \chi_g)$ are two eigenforms, then the motive $M_f \otimes M_g$ can only admit a Tate twist of weight -1 if $k+\ell$ is odd. This condition implies that $\chi_f\chi_g$ must be an odd Dirichlet character and hence cannot be trivial. In particular, the L-function $L(f \otimes g, s)$ does not have real coefficients and the tensor product $M_f \otimes M_g((k+\ell-1)/2)$ cannot be isomorphic to its Kummer dual, *unless* one of f or g is a CM form. In certain cases where g is a CM form occuring in the cohomology of $E^{\ell-1}$ for a suitable CM elliptic curve E, the so-called *generalised Heegner cycles* in $\mathrm{MW}(\mathcal{E}^r \times E^{\ell-1})$ that are studied in [BDP2] and reviewed in §5.2 seem to play the role of Heegner points and Heegner cycles for these motives.

4. Triple Rankin products. The main situation of interest to us in these notes (and, in particular, in the next chapter) arises from a triple (f_1, f_2, f_3) of modular forms of weights $k_1 \geq k_2 \geq k_3$ satisfying the Gross-Prasad hypothesis of Definition 6.2. In particular, in the balanced case where the sign ε in the functional equation for $L(f_1, f_2, f_3, s)$ is -1, the L-series $L(M_0(f_1, f_2, f_3), s)$ of the self-dual motive

$$M_0(f_1, f_2, f_3) := M_{f_1} \otimes M_{f_2} \otimes M_{f_3}((k_1+k_2+k_3-2)/2)$$

vanishes at $s = 0$. As an immediate consequence of Conjecture 6.3 we obtain the following which is reminiscent of Question 2.1.

Conjecture 6.4. *Assume (k_1, k_2, k_3) is a triple of balanced weights. Then there exists a null-homologous cycle of codimension $s_0 = (k_1+k_2+k_3-2)$ on $\mathcal{M}(f_1, f_2, f_3)$ whose class in $\mathrm{CH}^{s_0}(\mathcal{M}(f_1, f_2, f_3))_0$ is non-torsion.*

In the next chapter we will describe certain explicit null-homologous cycles on the triple of product of three Kuga-Sato varieties that may play the same role in the study of Conjecture 6.4 as Heegner points in the study of Question 2.1.

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Chapter 7

Diagonal cycles on the triple product of Kuga-Sato varieties

This chapter is devoted to studying the arithmetic of the self-dual motive $M_0(f_1, f_2, f_3)$ arising from the tensor product of three modular forms f_1, f_2 and f_3, from a perspective which is in harmony with the study carried out in Chapter 3. In particular, our goal is to construct null-homologous cycles in products of Kuga-Sato varieties playing the same role (for the motive $M_0(f_1, f_2, f_3)$) as

- Heegner points (explained in Chapter 3),
- Heegner cycles (studied in [Ne92]), and
- generalised Heegner cycles (considered in [BDP1] and [BDP2] and reviewed in §5.2)

played in the study of the arithmetic of the motive M_f associated with a newform of weight $k = 2$ (respectively of higher weight $k > 2$, the Rankin product $M_f \otimes M_g$ with g a modular form with complex multiplication.)

A particular instance of this problem was first considered by Gross, Kudla and Schoen in [7.6] and [7.7]. Namely, the authors take the weights of the three modular forms to be $k_1 = k_2 = k_3 = 2$, the levels to be all three the same $N = N_1 = N_2 = N_3$ and square-free, and the three characters to be trivial. As we review and generalize below in (7.3), when the Gross-Prasad hypothesis holds they construct a null-homologous cycle Δ_{GK} on the cube X^3 of the modular curve $X = X_0(N)$ and check some of its basic properties. They show for instance that Δ_{GK} extends in a suitable way to an integral model of X^3, and therefore that the Arakelov height $h(\Delta_{GK}(f_1, f_2, f_3))$ is indeed well-defined.

7.1 Diagonal cycles

For any non-negative integer $r \geq 0$, recall that $\mathcal{E}^r := \mathcal{E}_1^r(N)$ denotes the $(r+1)$-dimensional Kuga-Sato variety above $X_1(N)$ introduced in Chapter 5. It will be convenient to number the factors in $\mathcal{E}^r$, and to write

$$\mathcal{E}^r := \mathcal{E}_1 \times_{X_0(N)} \mathcal{E}_2 \times_{X_0(N)} \cdots \times_{X_0(N)} \mathcal{E}_r.$$

Choose three non-negative integers $r_1, r_2, r_3 \geq 0$ and define

$$V_{r_1,r_2,r_3} = \mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3}$$

to be the cartesian product of the three Kuga-Sato varieties $\mathcal{E}^{r_i}$, $i = 1, 2, 3$. It is variety of dimension $\sum_i (r_i + 1) = 2r + 3$. Similarly as above, the underlying variety of the motive $M(f_1, f_2, f_3)$ can be taken to be $V = V_{r_1,r_2,r_3}$. The remainder of this chapter is devoted to the construction of an explicit candidate $\Delta \in \mathrm{CH}^{r+2}(V)_0$ for the cycle alluded to in Conjecture 6.4. We do it by considering two separate cases. (See (7.2) and (7.3) for the precise definitions of the cycles in each chase.)

1. The case where $k_3 > 2$.

Lemma 7.1. *Suppose that A, B and C are subsets of $\{1, \ldots, r\}$ of cardinality r_1, r_2, and r_3 respectively, with*

$$r_1 + r_2 + r_3 = 2r.$$

Then the following conditions are equivalent:

1. *$A \cap B \cap C = \emptyset$;*
2. *Each element of $\{1, \ldots, r\}$ belongs to exactly two of the sets A, B and C.*
3. *$A \cup B = \{1, \ldots, r\}$, and C is the complement of $A \cap B$.*
4. *$A \cup C = \{1, \ldots, r\}$, and B is the complement of $A \cap C$.*
5. *$B \cup C = \{1, \ldots, r\}$, and A is the complement of $B \cap C$.*

Furthermore, if (A, B, C) and (A', B', C') are any two triples satisfying the above conditions, then there is a permuation $\sigma \in S_r$ satisfying

$$\sigma A = A', \quad \sigma B = B', \quad \sigma C = C'.$$

If the three sets A, B, C of cardinality r_1, r_2 and r_3 respectively satisfy the conditions in the lemma, we will say that (A, B, C) is a *cover* of $\{1, \ldots, r\}$.

Let

$$\begin{array}{rlcl} a: & \{1, \ldots, r_1\} & \longrightarrow & \{1, \ldots, r\}, \\ b: & \{1, \ldots, r_2\} & \longrightarrow & \{1, \ldots, r\}, \\ c: & \{1, \ldots, r_3\} & \longrightarrow & \{1, \ldots, r\} \end{array} \tag{7.1}$$

be three injective maps of sets, and let A, B and C denote the images of these maps. The data of (a, b, c) gives rise to a morphism which in the notation employed in Chapter 5 is defined by the rule

$$\begin{array}{rccc} \varphi_{a,b,c}: & \mathcal{E}^r & \longrightarrow & \mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3} \\ & (A, P_1, \ldots, P_r) & \mapsto & (A, P_{a(1)}, \ldots, P_{a(r_1)}), (A, P_{b(1)}, \ldots, P_{b(r_2)}), (A, P_{c(1)}, \ldots, P_{c(r_3)}). \end{array}$$

If we write e_i to denote the idempotent introduced in (5.11) acting on the factor $\mathcal{E}^{r_i}$, we can define

$$\Delta_{a,b,c} = (e_1, e_2, e_3)\varphi_{a,b,c}(\mathcal{E}^r).$$

As the reader can check, the cycle $\Delta_{a,b,c}$ is null homologous. In addition, there are good reasons to believe that cycle class $[\Delta_{a,b,c}] \in \mathrm{CH}^{r+2}(V)_0$ is *torsion* whenever the triple of sets $(\mathrm{Im}(a), \mathrm{Im}(b), \mathrm{Im}(c))$ is not a cover of $\{1, \dots, r\}$ in the above sense.

Set now $r_i := k_i - 2$ for $i = 1, 2, 3$ and $r = \frac{r_1+r_2+r_3}{2}$. The hypothesis made on the triple of weights (k_1, k_2, k_3) to be *balanced* can immediately be seen, by Lemma 7.1, to translate into the existence of sets A, B, and C of cardinality r_1, r_2, r_3, respectively, such that (A, B, C) is a cover of $\{1, \cdots, r\}$. Lemma 7.1 also ensures that such a cover is unique up to permutations. Choose injective maps α, β, γ as in (7.1) with images A, B and C. The above discussion leads us to hope that the cycle

$$\Delta_{f_1,f_2,f_3} := e_{f_1,f_2,f_3} \cdot \Delta_{\alpha,\beta,\gamma} \in \mathrm{CH}^{r+2}(V)[f_1, f_2, f_3]_0 = \mathrm{CH}^{r+2}(\mathcal{M}(f_1, f_2, f_3))_0 \qquad (7.2)$$

might sometimes be non-torsion, fullfilling the expectations raised in Conjecture 6.4, at least in some situations.

2. The case where $k_3 = 2$.

In this case, the fact that (k_1, k_2, k_3) is a balanced triple implies that (r_1, r_2, r_3) must be of the form $(r, r, 0)$ where $r \geq 0$. Choose the base point $O = (E_\infty, 1, \dots, 1) \in \mathcal{E}^r$ over the cusp ∞ on $X_0(N)$ corresponding to the generalized elliptic curve $E_\infty \simeq \mathbb{G}_m$.

Generalizing a construction of Gross, Kudla and Schoen introduced and studied in [7.6] and [7.7], we use the point $O \in \mathcal{E}^r$ to define cycle classes

$$\Delta_r \in \mathrm{CH}^{r+2}(\mathcal{E}^r \times \mathcal{E}^r \times X_0(N))_0$$

as follows. For any non-empty subset $I \subseteq \{1, 2, 3\}$, let

$$\Delta_I := \left\{ (P_1, P_2, P_3) \in \mathcal{E}^r \times \mathcal{E}^r \times X_0(N), \text{ such that } \begin{array}{l} P_i = P_j \text{ for all } i, j \in I, \\ P_k = O \text{ for all } k \notin I \end{array} \right\}$$

where, for any two points $P \in X_0(N)$ and $Q \in \mathcal{E}^r$, we write by an abuse of notation $P = Q$ to mean $P = \pi(Q)$.

Define $\Delta_{f_1,f_2,f_3} = e_{f_1,f_2,f_3} \cdot \Delta_r$ where

$$\Delta_r = \begin{cases} \Delta_{\{1,2,3\}} - \Delta_{\{1,2\}} - \Delta_{\{2,3\}} - \Delta_{\{1,3\}} + \Delta_1 + \Delta_2 + \Delta_3 & \text{if } r = 0 \\ (e_1, e_2, \mathrm{Id})(\Delta_{\{1,2,3\}} - \Delta_{\{1,2\}} & \text{if } r > 0. \end{cases} \qquad (7.3)$$

As before, we leave to the reader the task of checking that cycle Δ_r is null-homologous for all $r \geq 0$ and therefore that its class modulo rational equivalence yields an element $[\Delta_{f_1,f_2,f_3}]$ in $\mathrm{CH}^{r+2}(V)[f_1, f_2, f_3]_0$. Verifying that $[\Delta_{f_1,f_2,f_3}]$ is non-torsion is a highly non-trivial task. Following the philosophy of the Gross-Zagier formula discussed in Chapter 3, one way of proving such a statement would follow as a consequence of the following

Conjecture 7.2. *There exists a non-zero period* $\Omega_{f_1,f_2,f_3} \in \mathbb{C}^\times$ *such that*

$$\frac{\partial}{\partial s} L(f_1, f_2, f_3; s)_{s=s_0} = \Omega_{f_1,f_2,f_3} \cdot h(\Delta_{f_1,f_2,f_3}),$$

where h *stands for the canonical height pairing on* $\mathrm{CH}^{r+2}(V)_0$.

See the recent work of Yuan, S. Zhang and W. Zhang [7.12] for the last developments in this direction, in the particular scenario of weights $(k_1, k_2, k_3) = (2, 2, 2)$ that was already considered in [7.6] and [7.7].

7.2 Triple Chow-Heegner points

Our interest in these cycles is motivated by the questions discussed in the previous two chapters. Indeed, these cycles can be used to produce an interesting supply of Chow-Heegner points on an arbitrary elliptic curve $E/\mathbb{Q}$, *without having to rely* on the Hodge or Tate conjectures.

By way of comparison, recall from Chapter 5 that the authors of [BDP2] consider Chow-Heegner points on an elliptic curve E with complex multiplication arising from generalized Heegner cycles on the product $\mathcal{E}^r \times E^r$ of a Kuga-Sato variety by a suitable power of E. Cycles can be transferred to points on E by means of diagram (5.4), whose existence relies on the unproven validity of the Tate conjecture for $\mathcal{E}^r \times E^r$.

In the current scenario, when we choose the balanced triple of weights to be $(k, k, 2)$ with $k \geq 2$, the algebraic correspondence implicit in the diagram (5.4) is available thanks to the existence of the following natural cycle on $V_r \times X_0(N) = \mathcal{E}^r \times \mathcal{E}^r \times X_0(N) \times X_0(N)$.

Definition 7.3. Let Π be the $(r+2)$-dimensional variety $\mathcal{E}^r \times X_0(N)$, embedded diagonally in $\mathcal{E}^r \times \mathcal{E}^r \times X_0(N) \times X_0(N)$ by the map

$$\begin{array}{rccc} \iota = (\iota_{1,2}, \iota_{3,4}) : & \mathcal{E}^r \times X_0(N) & \longrightarrow & \mathcal{E}^r \times \mathcal{E}^r \times X_0(N) \times X_0(N) \\ & (x, y) & \mapsto & (x, x, y, y). \end{array} \tag{7.4}$$

The correspondence $\Pi \in \mathrm{CH}^{2r+2}(\mathcal{E}^r \times \mathcal{E}^r \times X_0(N) \times X_0(N))$ allows us to to transfer null-homologous cycles $\Delta \in \mathrm{CH}^{r+2}(\mathcal{E}^r \times \mathcal{E}^r \times X_0(N))_0$ to points P_Δ on $J_0(N) = \mathrm{CH}^1(X_0(N))_0$ by the recipe described in Chapter 5, which in this case is given by the rule

$$P_\Delta = \pi_{4,*}(\pi^*_{1,2,3}(\Delta) \cdot \Pi) = \sum Q \in J_0(N), \tag{7.5}$$

where

- a representative Δ has been chosen such that $\pi^*_{1,2,3}(\Delta)$ and Π meet transversaly.
- Q runs over the set of points (counted with multiplicity) of $X_0(N)$ for which there exists $P \in \mathcal{E}^r$ such that $(P, P, Q) \in \Delta$.

Of particular interest for us are the points

$$P_r = P_{\Delta_r} \in J_0(N)(\mathbb{Q}), \quad r \geq 0 \tag{7.6}$$

and

$$P_{f_1,f_2,f_3} = P_{\Delta_{f_1,f_2,f_3}} \in J_0(N)(\mathbb{Q}), \tag{7.7}$$

and their projection $\pi_E(P_r)$, $\pi_E(P_{f_1,f_2,f_3}) \in E(\mathbb{Q})$ to any elliptic curve E of conductor N by the modular parametrization 3.7, the former being a linear combination of the latter as (f_1, f_2, f_3) runs over triples of eigenforms of weights $(r+2, r+2, 2)$.

The computation of these points is however not straight-forward because the intersection of the two cycles $\pi_{1,2,3}^*\Delta_r$ and Π in $\mathcal{E}^r \times \mathcal{E}^r \times X_0(N) \times X_0(N)$ contains several irreducible components which do not meet transversally.

One particular instance in which an explicit, simple description of point $\pi_E(P_{f_1,f_2,f_3}) \in E(\mathbb{Q})$ can be given has been suggested by Zhang in [7.13]: if one takes $g := f_1 = f_2 \neq f := f_3 \in S_2(\Gamma_0(N))$ to be two eigenforms with *rational* fourier coefficients, they correspond to non-isogenous elliptic curves E_g and $E = E_f$ over $\mathbb{Q}$. Then

$$\pi_E(P_{g,g,f}) = \pi_{f,*}\pi_g^*(O_{E_g}) \in E(\mathbb{Q}). \tag{7.8}$$

In (7.8) and in the statement below, for any $n \in \mathbb{Z}$ use the isomorphism

$$\mathrm{Pic}^n(X_0(N)) \xrightarrow{\sim} \mathrm{Pic}^0(X_0(N)) \quad (\text{resp.} \, \mathrm{Pic}^n(E) \xrightarrow{\sim} \mathrm{Pic}^0(E))$$

induced by the map $D \mapsto D - n \cdot O$ on divisors to identify $\mathrm{Pic}^n(X_0(N))$ with $J_0(N) = \mathrm{Jac}(X_0(N))$, and $\mathrm{Pic}^n(E)$ with E.

One of the results obtained in [DRS] is the following; we refer to loc. cit. for more details and the proof.

Theorem 7.4. *Let K denote the canonical divisor on $X_0(N)$. Then*

$$P_0 = -K \in J_0(N).$$

Let $T_0 \subset J_0(N)(\bar{k})$ denote the (torsion) subgroup generated by divisors of degree zero supported at cusps in $X_0(N)$. For $r \geq 1$,

$$P_r = m_r P_0 \quad (\textit{mod } T_0).$$

for some (explicitly given) $m_r \in \mathbb{Z}$.

The reader may find instructive to compare some of the main results obtained in [BDP2] with the above quoted here. Indeed, Theorems 4 and 5 of [BDP2] constitute an analogue of Theorem 7.4 above in the context of generalized Heegner cycles and Chow-Heegner points (5.17) discussed in §5.2; notice though that in loc. cit. Theorem 5 is subject to the Hodge conjecture for the generalised Kuga-Sato variety (5.16) considered by the authors of [BDP2], and is only available unconditionally when one replaces the complex Abel-Jacobi map by its p-adic counterpart.

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Chapter 8

p-adic L-functions

Caveat. The last three chapters of these notes are quite rough and should be read in the spirit of an *informal discussion* of certain mathematical ideas. In particular, the exposition may be vague and even inaccurate in some places. These chapters will *not* be covered in the Arizona Winter School, and can therefore be ignored by the AWS participants. Our main motivation for including them here is that they provide a glimpse into some of the possible ramifications of the constructions that were discussed at the AWS and that the authors hope to pursue in the forthcoming work [DR].

8.1 p-adic families of motives

The theme of complex variation intervenes at a very basic stage in the definition of the L-value attached to a motive M in Section 6.2. Indeed, the complex L-series $L(M,s)$ gives a natural "complex interpolation" of the values $L(M(j))$ with $j \in \mathbb{Z}$ and $j >> 0$, and the very definition of $L(M)$ in general depends on the meromorphic continuation of $L(M,s)$.

Fixing a prime p, it is natural to consider p-adic families of motives—more precisely, of their p-adic étale realisations. To define this notion we begin with some general remarks about *Iwasawa algebras*, focussing for simplicity on the case where the prime p is *odd*.

In its most basic incarnation, the Iwasawa algebra Λ is the completed group ring $\mathbb{Z}_p[[G]]$, where $G = (1+p\mathbb{Z}_p)^\times \simeq \mathbb{Z}_p$. The natural inclusion $G \longrightarrow \mathbb{Z}_p^\times$ admits a canonical splitting $a \mapsto \langle a \rangle$, where $\langle a \rangle \in 1+p\mathbb{Z}_p$ is defined by

$$a = \omega(a)\langle a \rangle, \qquad \text{with } \omega(a) \in \mu_{p-1}(\mathbb{Z}_p).$$

The choice of a topological generator γ of G determines an isomorphism

$$\Lambda \simeq \mathbb{Z}_p[[T]], \qquad \gamma \mapsto 1+T.$$

The algebra Λ can also be identified with a subring of the ring of rigid analytic functions on the so-called *weight space*

$$\mathcal{X} := \hom_{\mathrm{cts}}(G, \mathbb{C}_p^\times)$$

via the natural pairing $\Lambda \times \mathcal{X} \longrightarrow \mathbb{C}_p$ given by extending $x \in \mathcal{X}$ continuously to Λ by $\mathbb{Z}_p$-linearity. Given $a \in \Lambda$, we will write

$$a(x) = x(a) := \langle a, x \rangle$$

interchangeably, depending on what notation is most suggestive.

A point $\nu \in \mathcal{X}$ is called an *arithmetic point* if there exists an integer $k \in \mathbb{Z}$ and a finite order Dirichlet character χ of p-power order and conductor with values in $\mathbb{C}_p^\times$ such that

$$\nu = \nu_{\chi,k}(x) := \chi(x)x^k, \qquad \text{for all } x \in (1 + p\mathbb{Z}_p).$$

The integer k, which is completely determined by the arithmetic point ν, is called the *weight* of this arithmetic point and we write $\mathrm{wt}(x) = k$.

More generally, a finite flat extension $\tilde{\Lambda}$ of Λ can be viewed geometrically as the space of rigid analytic functions on a finite covering $\tilde{\mathcal{X}} := \mathrm{Spec}(\tilde{\Lambda}) \longrightarrow \mathcal{X}$ of weight space. A point $x \in \tilde{\mathcal{X}}$ is then called an arithmetic point if it lies above an arithmetic point of $\mathcal{X}$, and the weight of x is defined in the obvious way. Of course, any $\nu \in \tilde{\mathcal{X}}$ can be viewed as a continuous ring homomorphism from $\tilde{\Lambda}$ to a finite extension E_ν of $\mathbb{Q}_p$ contained in $\mathbb{C}_p$.

Let $\mathbb{V}$ be a finitely generated, locally free $\tilde{\Lambda}$-module. The E_ν-vector space

$$\mathbb{V}_\nu := \mathbb{V} \otimes_{\tilde{\Lambda},\nu} E_\nu$$

is called the *specialisation of* $\mathbb{V}$ at the point ν.

Definition 8.1. A *p-adic family of motives* is a triple $\mathbb{M} = (\tilde{\Lambda}, \Sigma, \mathbb{V})$, where

- $\tilde{\Lambda}$ is a finite flat extension of the Iwasawa algebra Λ;
- Σ is a dense subset of the set of arithmetic points of $\tilde{\mathcal{X}}$;
- $\mathbb{V}$ is a locally free $\tilde{\Lambda}$-module of finite rank, equipped with a continuous $\tilde{\Lambda}$-linear action of $G_{\mathbb{Q}}$,

such that, for all $\nu \in \Sigma$, the specialisation $\mathbb{V}_\nu$ is the p-adic étale realisation of a motive M_ν over $\mathbb{Q}$. (More precisely, one requires that M_ν have coefficients in a field E_0 equipped with a homomorphism $E_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p \longrightarrow E$, for which

$$M_{\nu,p} \otimes_{E_0} E \simeq V_\nu.)$$

A p-adic family of motives is essentially a collection of motives $\{M_\nu\}_{\nu\in\Sigma}$ whose p-adic étale realisations vary analytically with ν and can therefore be extended to $\tilde{\mathcal{X}}$. The following are the most important and classically studied examples of p-adic families of motives.

1. Cyclotomic families.

The action of Galois on the p-power roots of unity gives rise to a natural continuous homomorphism (the so-called *cyclotomic character*)

$$G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_p^\times.$$

Composing this cyclotomic character with the projection $a \mapsto \langle a \rangle$ from $\mathbb{Z}_p^\times$ to G, followed by the natural inclusion of G into the set of "group-like elements" in the completed group ring Λ, leads to a Galois character

$$\varepsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \longrightarrow \Lambda^\times.$$

Let Λ_{cyc} be the free Λ-module of rank one equipped with the action of $G_{\mathbb{Q}}$ arising from $\varepsilon_{\mathrm{cyc}}$, and let Σ be the set of all arithmetic points of $\mathcal{X}$. For each $\nu = \nu_{k,\chi} \in \Sigma$, the specialisation of Λ_{cyc} at ν is the p-adic étale realisation of the Tate-Dirichlet motive $\mathbb{Z}(k)(\chi)$. The triple $(\Lambda, \Sigma, \Lambda_{\mathrm{cyc}})$ is the simplest non-trivial example of a p-adic family of motives.

More generally, if M is a fixed motive and $V = M_{p,\mathrm{et}}$ is its p-adic étale realisation, the representation

$$\mathbb{V}_{\mathrm{cyc}} := V \otimes_{\mathbb{Q}_p} \Lambda_{\mathrm{cyc}}$$

gives rise to a p-adic family of motives whose specialisation at $\nu_{k,\chi} \in \Sigma$ is the p-adic étale realisation of the motive $M(k)(\chi)$. The triple $M_{\mathrm{cyc}} := (\Lambda, \Sigma, \mathbb{V}_{\mathrm{cyc}})$ is called the *cyclotomic family* attached to M.

2. Hecke characters

The cyclotomic character $\varepsilon_{\mathrm{cyc}}$ is the prototypical example of a Hecke character of $\mathbb{Q}$. We may generalise the construction of the previous section by considering Hecke characters of an arbitrary number field K. For simplicity, we confine our attention in this section to the (already very interesting) special case where K is an imaginary quadratic field.

Suppose for simplicity that K is of class number one and let A be an elliptic curve over K with complex multiplication by $\mathcal{O}_K$. The p-adic Tate module of A is a free $\mathcal{O}_K \otimes \mathbb{Z}_p$-module of rank one, and the natural action of G_K on this module gives rise to a p-adic Galois character

$$\varepsilon_A : G_K \longrightarrow H := (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times.$$

The group H is a profinite group and can be written as a projective limit of finite quotients H/H_n where $\{H_n\}$ is a system of open normal subgroups of finite index in H such that $\cap_{n\geq 1} H_n = \{0\} \subset \cdots \subset H_n \subset H_{n+1} \subset \cdots \subset H$. Furthermore, H is isomorphic to $C \times H_p$ where C is a finite abelian group of exponent c relatively prime to p and H_p is equipped with an isomorphism

$$\gamma : \mathbb{Z}_p^2 \xrightarrow{\sim} H_p.$$

Let $\mathcal{O}$ denote the ring of integers of a finite extension of $\mathbb{Q}_p$ and let

$$\Lambda_H = \mathcal{O}[[H]] := \varprojlim \mathcal{O}[H/H_n]$$

denote the Iwasawa algebra of H. Let $\mathfrak{X}_H = \mathrm{Spf}(\Lambda_H)$ be the affine variety associated to Λ_H, and write

$$\mathrm{Hom}(H, \mathcal{O}^\times) \simeq \mathrm{Hom}(\Lambda_H, \mathcal{O}) = \mathfrak{X}_H(\mathcal{O}) \simeq \mathfrak{X}_C(\mathcal{O}) \times \mathfrak{X}_{H_p}(\mathcal{O})$$

for the group of continuous characters of H with values in $\mathcal{O}^\times$, which naturally extend to continuous ring homomorphisms between the Iwasawa algebra of H and $\mathcal{O}$ and can this way be regarded as the group $\mathfrak{X}_H(\mathcal{O})$ of $\mathcal{O}$-integral points of $\mathfrak{X}_H$. The Iwasawa algebra of H factors as

$$\Lambda_H \simeq \prod_{\psi \in \mathfrak{X}_C} \Lambda_{H_p}, \quad \Lambda_{H_p} \simeq \mathcal{O}[[T]] := \mathcal{O}[[T_1, \ldots, T_d]],\ h_i \mapsto T_i + 1.$$

By mimicking the construction of the previous section, with $\varepsilon_{\mathrm{cyc}}$ replaced by ε_A and G by H, we obtain a free module of rank one over Λ_H, denoted Λ_A, equipped with a continuous action of G_K via ε_A. Taking Σ to be the set of characters of H of the form

$$\nu_{k_1,k_2} : a \mapsto a^{k_1} \bar{a}^{k_2},$$

we find that the triple

$$M_A := (\Lambda_H, \Sigma, \Lambda_A) \tag{8.1}$$

is a p-adic family of motives, whose specialisation at the character ν_{k_1,k_2} is the p-adic étale realisation of an appropriate motive arising in the cohomology of $A^{k_1+k_2}$.

If one wishes to remain in the context of motives over $\mathbb{Q}$ rather than K, one can replace the one-dimensional representation Λ_A of G_K by its induced representation to $G_{\mathbb{Q}}$, which is now two-dimensional. We will often view M_A in this way as a p-adic family of CM motives of rank two over $\mathbb{Q}$.

3. Hida families

The richness of the notion of p-adic families of motives stems from the fact that there are other examples beyond the naturally arising families of Galois twists by cyclotomic and Hecke characters. These can be obtained from Hida's theory of p-adic families of (ordinary) eigenforms, which we quickly review.

Recall first that a classical normalised eigenform $g \in S_k(\Gamma_0(Np), \chi)$, with fourier coefficients in the ring of integers of a finite extension of $\mathbb{Q}_p$, is said to be *ordinary* at p if the p-th fourier coefficient $a_p(g)$ is a p-adic unit.

Definition 8.2. A Λ-adic modular form of tame level N and tame character χ is a pair $(\tilde{\Lambda}, \underline{g})$, where $\tilde{\Lambda}$ is a finite extension of Λ and

$$\underline{g} := \sum \underline{a}_n q^n, \qquad \underline{a}_n \in \tilde{\Lambda}$$

is a formal q-series with coefficients in $\tilde{\Lambda}$, such that, for almost all $x \in \tilde{\mathcal{X}}$ of weight $k \in \mathbb{Z}^{\geq 2}$, the power series

$$\underline{g}_x := \sum_{n=1}^{\infty} \underline{a}_n(x) q^n$$

is the q-expansion of a classical normalised ordinary eigenform of weight k and level N.

A point $x \in \tilde{\mathcal{X}}$ for which the specialisation $\underline{g}_x$ is the q expansion of a classical modular form (of some weight and level) will sometimes be referred to as a *classical point* of $\tilde{\mathcal{X}}$, and the set of all classical points will be denoted Σ_{cl}. (Of course, this notion of classicality is not an intrinsic feature of x as an element of $\tilde{\mathcal{X}}$, but rather depends on the family $\underline{g}$ of modular forms that is being parametrised by $\tilde{\mathcal{X}}$.)

Some classical examples of p-adic families of modular forms are given by

(a) Eisenstein series of varying weights,

$$E_k := \zeta^*(1-2k) + 2\sum_{n=1}^{\infty} \sigma^*_{2k-1}(n)\, q^n,$$

where $\zeta^*(s) = (1-p^{-s})\zeta(s)$ is the Riemann zeta function with its Euler factor at p removed, and $\sigma^*_{2k-1}(n) = \sum_{\substack{d|n \\ (p,d)=1}} d^{2k-1}$.

(b) The binary theta series associated to the powers of a fixed Hecke Grossencharacter Ψ of infinity type $(1,0)$ of an imaginary quadratic field K. These theta series are defined by letting, for all ideals $a \lhd \mathcal{O}_K$ of the ring of integers of K,

$$\Psi^*(a) = \begin{cases} \Psi(a) & \text{if } p \nmid a\bar{a}, \\ 0 & \text{otherwise,} \end{cases}$$

and setting

$$\theta_k := \sum_{a \lhd \mathcal{O}_K} \Psi^*(a)^{2k-1}\, q^{a\bar{a}}. \tag{8.2}$$

The fact that Λ-adic eigenforms exist in even greater abundance can be seen from the following theorem of Hida (cf. for instance Theorem 3 of Section 7.3. of [Hi93]) which will play a key role in the construction of the last chapter.

Theorem 8.3 (Hida). *Let $g \in S_k(\Gamma_1(Np))$ be an ordinary normalised eigenform of weight $k \geq 1$. Then there exists a Λ-adic eigenform $(\tilde{\Lambda}_g, \underline{g})$ of tame level N and a weight k point $x \in \tilde{\mathcal{X}}$ such that $\underline{g}_x = g$.*

We then say that the Λ-adic eigenform $\underline{g}$ *interpolates g in weight k*. When $k \geq 2$, it is known that the Λ-adic form $\underline{g}$ interpolating g in weight k is *unique*. This uniqueness fails in general in the (perhaps most interesting) case where $k = 1$.

Given a Λ-adic family $\underline{g}$ of modular forms, one can consider the Serre-Deligne p-adic representations V_{g_ν} attached to the specialisations of $\underline{g}$ at the classical points $\nu \in \Sigma$. The following theorem of Hida asserts that these p-adic Galois representations also interpolate to a p-adic family.

Theorem 8.4 (Hida). *Let $(\tilde{\Lambda}, \underline{g})$ be an ordinary $\tilde{\Lambda}$-adic modular form. Then there exists a p-adic family of motives of the form $\mathbb{M}_g := (\tilde{\Lambda}, \Sigma_{\mathrm{cl}}, \mathbb{V})$ such that*

$$V_\nu \simeq V_{g_\nu}, \qquad \textit{for all } \nu \in \Sigma_{\mathrm{cl}}.$$

The theory of p-adic families of modular forms and their associated Λ-adic representations thus gives rise to interesting examples of p-adic families of Deligne-Scholl motives attached to modular forms.

In the special case where $\underline{g}$ is a CM family of modular forms as described in equation (8.2) above, one recovers the family of motives arising from Hecke characters of K described in (8.1), viewed as rank two motives over $\mathbb{Q}$. The p-adic families of motives arising from Hida's theory are a far-reaching generalisation of these basic examples. We remark in passing that, unlike what happens with cyclotomic families or families arising from Hecke characters, the Λ-adic representations arising from non-CM Hida families often have large image which can even be open in $\mathbf{GL}_2(\tilde{\Lambda})$.

8.2 p-adic L-functions

All p-adic L-functions of arithmetic interest arise from p-adic families of motives of the form $\mathbb{M} := (\tilde{\Lambda}, \Sigma, \mathbb{V})$ by considering the assignment

$$\nu \mapsto L(M_\nu), \qquad \nu \in \Sigma,$$

where $L(M_\nu)$ is the L-value attached to the motive M_ν as in Section 6.2. Naively, one could ask whether it is possible to extend this function on Σ to a p-adic analytic function on $\tilde{\mathcal{X}}$.

There are several obvious difficulties that one faces in trying to make this precise. Most importantly, the L-values $L(M_\nu)$ (assuming they can even be defined, which often relies on the analytic continuation of $L(M_\nu, s)$) are complex numbers, and there is no natural way to view them as p-adic numbers.

There are several ways one can remedy this difficulty:

1. The most traditional approach is to consider the subset $\Sigma_{\mathrm{crit}} \subset \Sigma$ consisting of points ν for which the motive M_ν is *critical* in the sense of Section 6.2. After dividing $L(M_\nu)$ by the suitable Deligne period Ω_{M_ν}, and removing the Euler factor at p, one obtains algebraic numbers

$$L_{\mathrm{alg}}(M_\nu) := L(M_\nu)/\Omega(M_\nu), \qquad \mathscr{L}_p(M_\nu) := P_{M_\nu,p}(1) L_{\mathrm{alg}}(M_\nu), \tag{8.3}$$

where $P_{M_\nu,p}(1)$ is the Euler factor that appears in the definition (6.2) of $L(M_\nu, s)$ at $s = 0$. The expressions in (8.3) can therefore be viewed as p-adic numbers after fixing an inclusion $\bar{\mathbb{Q}} \longrightarrow \bar{\mathbb{Q}}_p$. It then becomes legitimate to try to interpolate them p-adically. Of course, the question of p-adic interpolation presupposes the ability to make a coherent integral choice of the periods $\Omega(M_\nu)$ as ν varies. The description of such canonical integral periods in the case of the motive attached to the Rankin triple product of three modular forms is the theme of Prasanna's lectures at the AWS; we will say very little here about this rather delicate question.

2. In some cases (for example, in cases where the family $\mathbb{M}$ does not admit sufficiently many critical specialisations) one may try to resort to other algebraic interpretations

of $L(M_\nu)$, such as those given by the Bloch-Beilinson conjectures. These conjectures express $L(M_\nu)$ in terms of regulators evaluated on elements of algebraic K-theory or Chow groups. It makes no sense to attempt to directly interpolate the regulators, which are (ostensibly transcendental) complex numbers. However, one could envisage putting the algebraic objects which conjecturally "account for" the values $L(M_\nu)$ into p-adic families. To make this idea (a bit) more precise, recall the statement of the Bloch-Beilinson conjectures given in Section 6.3. Suppose Σ consists of a set of arithmetic points for which the specialisations M_ν are all self -dual in the sense of that section, and for which $L(M_\nu)$ vanishes identically. One might then hope for a systematic construction of elements $P_\nu \in \mathrm{MW}(M_\nu)$ akin to Heegner points, whose heights (in the Arakelov sense) should encode $L'(M_\nu, 0)$. The image of P_ν under the p-adic étale Abel-Jacobi map belongs to the continuous Galois cohomology groups $H^1(\mathbb{Q}, M_{p,\nu})$. One would thus obtain (assuming the availability of the system P_ν) a natural collection of cohomology classes $\kappa_\nu \in H^1(\mathbb{Q}, M_\nu)$. It makes sense to try to interpolate these classes p-adically, i.e., to construct a class $\underline{\kappa} \in H^1(\mathbb{Q}, \mathbb{V})$ from which the classes κ_ν are obtained by specialisation. One can then hope to extract the p-adic L-function $\mathscr{L}_p(\mathbb{M})$ by considering various p-adic invariants attached to $\underline{\kappa}$, such as its image under a dual exponential map studied by Kato and Perrin-Riou.

We will not say anything about the second approach, other than to note its potential usefulness in shedding light on the Stark-Heegner point constructions that are detailed in the last chapter of these notes.

From now on, we will confine our attention to the more traditional approach to p-adic L-functions based on critical values in the sense of Deligne. The following rather loose definition assumes a sufficiently rigid and well-behaved definition of the Deligne periods $\Omega(M_\nu)$ and of the "algebraic parts" $L_{\mathrm{alg}}(M_\nu)$ as ν varies.

Definition 8.5. The p-adic L-function attached to a p-adic family $\mathbb{M} = (\tilde{\Lambda}, \Sigma, \mathbb{V})$ of motives is a pair $(\Sigma_L, \mathscr{L}_p(\mathbb{M}))$, where

1. $\Sigma_L \subset \Sigma_{\mathrm{crit}}$ is a subset of the set of critical points in Σ, which is dense in $\tilde{\mathscr{X}}$;

2. $\mathscr{L}_p(\mathbb{M})$ is an element of $\tilde{\Lambda}$,

such that

$$\mathscr{L}_p(\mathbb{M})(\nu)(:= \nu(\mathscr{L}_p(\mathbb{M}))) = \mathscr{L}_p(M_\nu), \qquad \text{for all } \nu \in \Sigma_L,$$

where $\mathscr{L}_p(M_\nu)$ is given in (8.3).

The Iwasawa algebra of H can also be interpreted as the space $\mathrm{Meas}(H, \mathcal{O})$ of measures on the topological group H with values in $\mathcal{O}$, which in turn can be seen as a projective limit of step functions on the finite sets H/H_n; given $\alpha \in \Lambda_H$ we may write $\mu_\alpha \in \mathrm{Meas}(H, \mathcal{O})$ for the corresponding measure, and given a character $\chi \in \mathfrak{X}_H$ we write

$$\int_H \chi d\mu_\alpha := \mu_\alpha(\chi).$$

The following are the most important basic examples of p-adic L-functions.

1. The Kubota-Leopoldt p-adic L-function.

Let χ be a classical even Dirichlet character and let $\mathbb{M}(\chi) = (\Lambda, \Sigma, \Lambda_{\mathrm{cyc}}(\chi))$ be the p-adic family of motives associated to the Dirichlet motive $\mathbb{Z}(\chi)$. Then, letting

$$\Sigma_L = \{\nu_k \in \Sigma \quad \text{with } k \in \mathbb{Z}^{\leq 0}, \}$$

there exists $\mathscr{L}_p(\chi) \in \Lambda$ for which

$$\nu_k(\mathscr{L}_p(\chi)) = (1 - \chi(p)\omega(p)^{-k}p^{-k})L(k, \chi\omega^{-k}), \qquad \text{for all } k \in \Sigma_L.$$

(Cf. Theorem 2 of [8.6].) In this case, where one interpolates special values of Dirichlet L-functions at the *negative* integers, the period $\Omega(M_\nu)$ can be taken to be equal to 1. The element $\mathscr{L}_p(\chi)$ attached to the family $\mathbb{M}(\chi)$ is called the *Kubota-Leopoldt p-adic L-function* of χ. For more details on this construction, see the classic book [8.6] of Iwasawa.

2. The Katz p-adic L-function The work of Katz (cf. [8.7]) likewise associates a p-adic L-function to a p-adic family of CM motives arising from Hecke characters.

Let ν be a Hecke character of the imaginary quadratic field K, of infinity type (k_1, k_2) and conductor $\mathfrak{f}$, and define its L-series as

$$L(\nu, s) = \sum_{\mathfrak{a}} \nu(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} (1 - \nu(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1},$$

where the sum is taken over the integral ideals of K and the product over the prime ideals in this number field. It extends to a meromorphic function on the whole complex plane and satisfies a functional equation. Namely, writing $\Lambda(\nu, s) = (D\mathrm{N}(\mathfrak{f})^{s/2})(2\pi)^{\min(k,j)-s}\Gamma(s - \min(k_1, k_2))L(\nu, s)$, we have

$$\Lambda(\nu, s) = \omega(\nu)\Lambda(\bar{\nu}, 1 + k_1 + k_2 - s) \tag{8.4}$$

where $\omega(\nu) \in \mathbb{C}$, $|\omega(\nu)| = 1$ is the so-called root number of ν.

Assume p splits in K, denote by $\mathfrak{p}$ the prime above p corresponding to the chosen embedding $K \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and fix an integral ideal $\mathfrak{c}$ of K which is prime to p. Let $\Sigma(\mathfrak{c})$ denote the set of all Hecke characters of K of conductor dividing $\mathfrak{c}$.

A character $\nu \in \Sigma(\mathfrak{c})$ is called *critical* if $L(\nu^{-1}, 0)$ is a critical value in the sense of Deligne, i.e., if the Γ-factors that arise in both sides the functional equation (8.4) of $\Lambda(\nu^{-1}, s)$ are non-vanishing and have no poles at $s = 0$. The set $\Sigma_{\mathrm{crit}}(\mathfrak{c})$ of critical characters can be expressed as the disjoint union

$$\Sigma_{\mathrm{crit}}(\mathfrak{c}) = \Sigma_{\mathrm{crit}}^{(1)}(\mathfrak{c}) \cup \Sigma_{\mathrm{crit}}^{(2)}(\mathfrak{c}),$$

where

$$\Sigma_{\mathrm{crit}}^{(1)}(\mathfrak{c}) = \{\nu \in \Sigma(\mathfrak{c}) \quad \text{of type } (k_1, k_2) \text{ with } k_1 \leq 0, \quad k_2 \geq 1\},$$

$$\Sigma_{\mathrm{crit}}^{(2)}(\mathfrak{c}) = \{\nu \in \Sigma(\mathfrak{c}) \quad \text{of type } (k_1, k_2) \text{ with } k_1 \geq 1, \quad k_2 \leq 0\}.$$

Assume $\mathfrak{c} = \bar{\mathfrak{c}}$ and let $*$ denote the involution on $\Sigma(\mathfrak{c})$ defined as $\nu^* := \nu \circ c$, where c denotes complex conjugation on the ideals of K. It is readily seen by comparing Euler factors that $L(\nu, s) = L(\nu^*, s)$. The regions $\Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c})$ and $\Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$ are interchanged by $*$.

Let F/K be a finite extension of K that contains the values $\nu(\mathfrak{a})$ as ν ranges over all Hecke characters in $\Sigma_{\mathrm{crit}}(\mathfrak{c})$ and $\mathfrak{a}$ ranges over $\mathbb{A}^\times_{K,f}$, the finite idèles of K. Choose a prime $\wp_F$ above $\wp$ and write $\mathbb{A}^{\times(p)}_{K,f}$ for the subgroup of $\mathbb{A}^\times_{K,f}$ of idèles whose local components at the primes $\wp$, $\bar{\wp}$ of K above p are integral units. The restriction of any $\nu \in \Sigma_{\mathrm{crit}}(\mathfrak{c})$ at $\mathbb{A}^{\times(p)}_{K,f}$ takes values in F which are locally integral at $\wp_F$ and this allows us to regard $\Sigma_{\mathrm{crit}}(\mathfrak{c})$ within the space $F_{\mathrm{an}}(\mathbb{A}^{\times(p)}_{K,f}, \mathcal{O}_{F_{\wp_F}})$ of analytic functions on $\mathbb{A}^{\times(p)}_{K,f}$ with values on $\mathcal{O} := \mathcal{O}_{F_{\wp_F}}$, which is equipped with a p-adic natural compact open topology. Both the subsets $\Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c})$ and $\Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$ are dense in the completion $\hat{\Sigma}_{\mathrm{crit}}(\mathfrak{c})$ relative to this topology.

Let $W = \mathbb{Z}[\mathrm{Hom}(K, \mathbb{C})] \simeq \mathbb{Z}^2$ denote the module of infinity types of Hecke characters of K (or module of weights of $G = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$, in the notation of §8). The results of Katz in [8.7] show that $\hat{\Sigma}_{\mathrm{crit}}(\mathfrak{c})$ can be interpreted as the set of $\mathcal{O}$-integral points of a rigid analytic variety $\mathcal{X}_\nu = \mathrm{Spf}(\Lambda_\nu)$ associated to a finite extension Λ_ν of the Iwasawa algebra Λ_W of $W \otimes \mathcal{O}$. As expected, the restriction of the resulting weight map $w : \hat{\Sigma}_{\mathrm{crit}}(\mathfrak{c}) = \mathcal{X}_\nu(\mathcal{O}) \longrightarrow W(\mathcal{O})$ to $\Sigma_{\mathrm{crit}}(\mathfrak{c})$ is given by the rule $\nu \mapsto w(\nu) = (k_1, k_2)$, the infinity type of ν.

Theorem 8.6. *(Katz [8.7]) There exists an element $\mathscr{L}_p^{Katz} \in \mathbb{C}_p \otimes \Lambda_\nu$ inducing a p-adic L-function*

$$\mathscr{L}_p^{Katz} : \hat{\Sigma}_{\mathrm{crit}}(\mathfrak{c}) \longrightarrow \mathbb{C}_p,$$

determined by the interpolation property:

$$\frac{\mathscr{L}_p^{Katz}(\nu)}{\Omega_p^{k_1-k_2}} = \left(\frac{\sqrt{D}}{2\pi}\right)^{k_2} (k_1-1)!(1-\nu(\mathfrak{p})/p)(1-\nu^{-1}(\bar{\mathfrak{p}}))\frac{L_{\mathfrak{c}}(\nu^{-1}, 0)}{\Omega^{k_1-k_2}}, \tag{8.5}$$

for all critical characters $\nu \in \Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$ of infinity type (k_1, k_2).

In (8.5), $\Omega \in \mathbb{C}$ and $\Omega_p \in \mathbb{C}_p$ are transcendental periods which ensure that both sides of the equality belong to $\bar{\mathbb{Q}}$.

Remark 8.7. Note that, despite $L(\nu, s) = L(\nu^*, s)$ for all $\nu \in \Sigma(\mathfrak{c})$, a similar equality need not hold in $\hat{\Sigma}(\mathfrak{c})$, because the involution $*$ induces the map $(k_1, k_2) \mapsto (k_2, k_1)$ on $W(\mathbb{Z}_p)$ and therefore does not preserve the lower right quadrant of weights of Hecke characters that lie in the range of classical interpolation. In particular, if $\nu \in \Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$, then $L_p^{Katz}(\nu^*)$ is an *a priori* mysterious p-adic avatar of $L(\nu^{-1}, 0)$. A formula of Rubin, which will be briefly alluded to in the next chapter, relates it to the p-adic logarithms of rational points on the associated elliptic curve.

3. The Mazur-Swinnerton-Dyer p-adic L-function. Let E be a (modular) elliptic curve over $\mathbb{Q}$ and let

$$\mathbb{M} = h^1(E)_{\mathrm{cyc}}$$

be the cyclotomic family associated to the motive $h^1(E)$, whose underlying Λ-adic Galois representation is

$$\mathbb{V} := H^1_{\mathrm{et}}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_p) \otimes \Lambda_{\mathrm{cyc}}.$$

Then the only critical points among the set Σ of arithmetic points in $\mathcal{X}$ are those of the form $\nu_{k,\chi}$, where $k = 1$ and χ is a Dirichlet character of p-power conductor. The set Σ_{crit} of all such $\nu_{1,\chi}$ is not dense in $\mathcal{X}$ for the natural p-adic topology–it corresponds essentially to the set of p-power roots of unity in the open unit disc centered at 1 in $\mathbb{C}_p$. However, by the Weierstrass preparation theorem, any element in the Iwasawa algebra $\mathbb{Z}_p[[G]]$ is completely determined by its values on Σ_{crit}.

Assume that the elliptic curve E is ordinary at p, i.e., that the polynomial $x^2 - a_p(E)x + p$ has a root, α_p, which is a p-adic unit. Then Mazur and Swinnerton-Dyer have shown the existence of an element $\mathscr{L}_p(E) \in \Lambda \otimes \mathbb{Q}$ interpolating the values

$$L(\nu_{1,\chi}(h^1(E)_{\mathrm{cyc}})) = L(E, \chi, 1).$$

This construction extends to the case where $h^1(E)$ is replaced by the Deligne-Scholl motive M_f of an arbitrary modular form of weight $k \geq 2$, which is ordinary at p. In that case one can take the set Σ_L of critical values of interpolation to be the set of $\nu_{j,\chi}$ with $1 \leq j \leq k-1$. The resulting p-adic L-function $\mathscr{L}_p(f)$, which belongs to Λ, is called the *Mazur-Swinnerton-Dyer p-adic L-function* attached to the modular form f. For the details on the definition of this p-adic L-function see [8.9] for example.

4. The Mazur-Kitagawa p-adic L-function.

Let f be a normalised newform of weight $k_0 \geq 2$ (arising, for example, from an elliptic curve E) which is ordinary at p. Hida theory gives rise to a p-adic family $\{f_\nu\}$ of modular forms, parametrized by $\nu \in \tilde{\mathcal{X}}$, and interpolating f in weight k_0. This in turn gives rise to a p-adic family of motives of the form

$$\mathbb{M}_f = (\tilde{\Lambda}, \Sigma_f, \mathbb{V}_f)$$

whose specialisations $\mathbb{M}_{f,\nu}$ at any point $\nu \in \Sigma_f$ is the Deligne-Scholl motive attached to f_ν. The one-variable family M_ν can be extended to a two-variable family

$$\mathbb{M}_{f,\mathrm{cyc}} = (\tilde{\Lambda} \otimes_{\mathbb{Z}_p} \Lambda, \Sigma_f \times \Sigma, \mathbb{V}_f \otimes \mathbb{Z}_p \Lambda_{\mathrm{cyc}})$$

by incorporating a cyclotomic variable. The specialisation of $M_{f,\mathrm{cyc}}$ to the point $(k, j) \in \Sigma_f \times \Sigma$ is just the motive $M_{f_k}(j)$.

The p-adic L-function $\mathscr{L}_p(E) \in \tilde{\Lambda} \otimes \Lambda$ attached to this two variable family is called the *Mazur-Kitagawa p-adic L-function* of the modular form f. For a detailed description of its construction and the precise interpolation property that it satisfies, see the exposition in [8.2] or in [8.8].

5. p-adic Rankin L-functions

If f and g be two normalised cuspidal eigenforms which are ordinary at p, and let $(\tilde{\Lambda}_f, \underline{f})$ and $(\tilde{\Lambda}_g, \underline{g})$ be the associated Hida families. A triple $(x, y, m) \in \Sigma_f \times \Sigma_g \times \Sigma$ is said to be *critical* if $\mathrm{wt}(x)$, $\mathrm{wt}(y)$, belong to $\mathbb{Z}^{\geq 2}$ and m is an integer with

$$\mathrm{wt}(x) \leq m \leq \mathrm{wt}(y) - 1.$$

In Theorem 1 of Section 10.4 of [Hi93], Hida constructs a *three*-variable p-adic L-function in $\tilde{\Lambda}_f \otimes \tilde{\Lambda}_g \otimes \Lambda$, denoted $\mathscr{L}_p(\underline{f}, \underline{g}, m)$ and satisfying an interpolation property of the type

$$\mathscr{L}_p(\underline{f}_x, \underline{g}_y, m) \sim \frac{L(\underline{f}_x \otimes \underline{g}_y, m)}{\langle \underline{g}_y, \underline{g}_y \rangle}, \qquad \text{for all critical } (x, y, m) \in \Sigma_f \times \Sigma_g \times \Sigma,$$

where the symbol $\sim$ denotes equality up to a simple and (at least in principle) completely explicit fudge factor.

Remark 8.8. Although the modular forms f and g play completely symmetrical roles in the definition of the complex Rankin convolution L-series $L(f \otimes g, s)$, this is not the case for its p-adic analogues. The p-adic L-functions $\mathscr{L}_p(\underline{f}, \underline{g})$ and $\mathscr{L}_p(\underline{g}, \underline{f})$ are defined by completely different interpolation properties, and bear no obvious relation to one another. The availability of two fundamentally distinct p-adic L-functions attached to the p-adic family of motives $\mathbb{M}_f \otimes \mathbb{M}_g$ is one of the features which make the p-adic world particularly rich, and it will play a key role in the discussions in the next two chapters.

6. Triple product p-adic Rankin L-functions

Let f, g, and h be three normalised cuspidal eigenforms which are ordinary at p, and let $(\tilde{\Lambda}_f, \underline{f})$, $(\tilde{\Lambda}_g, \underline{g})$ and $(\tilde{\Lambda}_h, \underline{h})$ and be the associated Hida families. One is particularly interested in the three variable family of self-dual motives given by

$$M_*(f_k, g_\ell, h_m) := M_{f_k} \otimes M_{g_\ell} \otimes M_{h_m}((k + \ell + m - 2)/2).$$

In [8.3], there is a discussion of a p-adic L-function $\mathscr{L}_p(f, g, h)$ interpolating the classical special values

$$L(M_*(f_k, g_\ell, h_m)) = L(f_k \otimes g_\ell \otimes h_m, (k + \ell + m - 2)/2), \qquad \text{where } k \geq \ell + m.$$

It is interesting to note that the region Σ_f of classical interpolation which defines this p-adic L-function is not invariant under interchanging f, g and h. So the construction of [8.3] really leads to *three distinct* p-adic L-functions $\mathscr{L}_p(f, g, h)$, $\mathscr{L}_p(g, h, f)$ and $\mathscr{L}_p(h, f, g)$ defined by quite different p-adic interpolation properties. If (k, ℓ, m) is any triple of classical weights, the values

$$\mathscr{L}_p(f, g, h)(k, \ell, m), \qquad \mathscr{L}_p(g, h, f)(\ell, m, k), \qquad \mathscr{L}_p(h, f, g)(m, k, \ell)$$

constitute three *a priori* quite different p-adic avatars of the central critical value

$$L(f_k \otimes g_\ell \otimes h_m, (k + \ell + m - 2)/2).$$

For further discussion of p-adic L-functions attached to Hida families, the reader is referred to [MTT], [8.4], [8.2], and [8.8]. We also refer to the notes by R. Pollack and G. Stevens in this Winter School for more details about the construction and practical calculation of these p-adic L-functions using the theory of *overconvergent modular symbols*. Finally, the reader may also consult [8.1], [8.10], [8.4], and [8.5] for other scenarios.

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Chapter 9

Two p-adic Gross-Zagier formulae

The purpose of this chapter is to illustrate the theory of p-adic L-functions and their special values by describing two distinct p-adic analogues of the Gross-Zagier formula of Chapter 3.

For the sake of simplicity, we place ourselves in the simplest setting for a Gross-Zagier formula, namely, the setting that was originally treated in [GZ86].

More precisely, we assume that E is an elliptic curve over $\mathbb{Q}$ of conductor N, and that K is a quadratic imaginary field of odd discriminant D in which all the prime divisors of N are split. Let $f \in S_2(\Gamma_0(N))$ be the eigenform of weight 2 attached to E, and let $\theta \in M_1(\Gamma_0(D), \varepsilon_K)$ be the theta series of weight one attached to the trivial character of K. The classical Gross-Zagier formula is concerned with the central critical value $L(f \otimes \theta, 1)$. In the situation where the Heegner hypothesis is satisfied, considerations of signs in functional equations as in (3.2) force $L(f \otimes \theta, s)$ to vanish at $s = 1$. The Gross-Zagier formula in Theorem 3.4 of Chapter 3 then expresses $L'(f \otimes \theta, 1)$ in terms of the height of a canonical Heegner point $P_K \in E(K)$. We recall that this point is the image of certain CM divisors on $X_0(N)$ under the modular parametrisation $X_0(N) \longrightarrow E$.

In striving for a p-adic analogue of the Gross-Zagier formula, the Rankin L-function $L(f \otimes \theta, s)$ is replaced by one of its p-adic analogues. In the p-adic setting, the situation is enriched by the fact that, as explained in Remark 8.8, there are, at the outset, *two natural candidates* for such a p-adic analogue.

More precisely, assume that E is ordinary at p, so that the eigenform f is the weight two specialisation of a unique Hida family $\underline{f}$, and let $\mathbb{M}_f$ be the associated p-adic family of Deligne-Scholl motives. The imaginary quadratic field K gives rise, likewise, to a p-adic family of Hecke characters by fixing a basic Hecke character ψ of infinity type $(1, 0)$ and trivial central character, and considering the powers $\psi^{\ell-1}$ of this Hecke character. Assuming that the prime p is split in K, one can see (as explained in Section 8.1) that the theta series $\theta_\ell := \theta_{\psi^{\ell-1}}$ is the weight ℓ specialisation of a Hida family $\underline{\theta}$ of CM forms interpolating θ_1 in weight one.

We can thus consider the two different p-adic Rankin L-functions $\mathscr{L}_p(\underline{f}, \underline{\theta})$ and $\mathscr{L}_p(\underline{\theta}, \underline{f})$ attached to the three variable p-adic family $\mathbb{M}_f \otimes \mathbb{M}_\theta \otimes \Lambda_{\mathrm{cyc}}$ of motives. They are both p-adic analytic functions of three p-adic variables (k, ℓ, j) and interpolate the classical special values

$L(f_k \otimes \theta_\ell, j)$ in the ranges

$$1 \le \ell \le j \le k-1 \qquad \text{and} \qquad 1 \le k \le j \le \ell - 1$$

respectively. In the setting of the p-adic Gross-Zagier formula, one is particularly interested in the behaviour of these functions at the point $\nu_{2,1,1}$ corresponding to $(k, \ell, j) = (2, 1, 1)$. This point is in the range of classical interpolation for $\mathscr{L}_p(\underline{f}, \underline{\theta})$ and therefore

$$\mathscr{L}_p(\underline{f}, \underline{\theta})(\nu_{2,1,1}) = 0.$$

The following theorem of Perrin-Riou considers the first derivative of this p-adic L-function at $\nu_{2,1,1}$ in the direction of the third (cyclotomic) variable.

Theorem 9.1 (9.3). *The derivative in the "cyclotomic direction" of $\mathscr{L}_p(\underline{f}, \underline{\theta})$ at the point $\nu_{2,1,1}$ is given by*

$$\frac{d}{ds}\mathscr{L}_p(\underline{f}, \underline{\theta})(\nu_{2,1,s})_{|s=1} = \langle P_K, P_K \rangle \pmod{K_f^\times}, \tag{9.1}$$

where $\langle\ ,\ \rangle$ denoted the canonical bilinear cyclotomic p-adic height pairing

$$\langle\ ,\ \rangle : J_0(N)(H_c) \otimes_{\mathbb{Z}} J_0(N)(H_c) \longrightarrow \bar{\mathbb{Q}}_p.$$

We refer to [9.2] for the generalisation of this result to modular forms f of arbitrary even weight k, in which Heegner points need to be replaced by Heegner divisors in Chow groups of Kuga-Sato varieties.

The point $\nu_{2,1,1}$ corresponding to $(k, \ell, j) = (2, 1, 1)$ lies outside the range of p-adic interpolation defining the second p-adic L-function $\mathscr{L}_p(\underline{\theta}, \underline{f})$, and hence the vanishing of $L(f \otimes \theta_1, 1)$ does not force the vanishing of $\mathscr{L}_p(\underline{\theta}, \underline{f})$ at $\nu_{1,2,1}$. In fact, this special value is non-zero precisely when $L'(f \otimes \theta_1, 1) \neq 0$, as the following result shows:

Theorem 9.2 ([BDP1]). *The value of $\mathscr{L}_p(\underline{\theta}, \underline{f})$ at the central critical point attached to the weight two specialisation of $\underline{f}$ and the weight one specialsation of $\underline{\theta}$ is given by*

$$\mathscr{L}_p(\underline{\theta}_1, \underline{f}_2) = \log^2_{\omega_E}(P_K)^2 \pmod{K_f^\times}, \tag{9.2}$$

where $\log_{\omega_E}$ denotes the p-adic formal group logarithm on $E(K)$ attached to a regular differential $\omega_E \in \Omega^1(E/\mathbb{Q})$.

The article [BDP1] also contains a more general version expressing the values of

$$\mathscr{L}_p(\underline{\theta}_\ell, \underline{f}_k) \quad \text{with } \ell < k$$

in terms of the images of certain "generalised Heegner cycles" under the p-adic Abel-Jacobi map.

Theorem 9.2 is a prototypical example of a situation where the the numerical evaluation of p-adic L-series attached to an elliptic curve E can be used to recover points of infinite order on E.

The first example of a formula of this type was discovered by Karl Rubin in [Ru], and concerns the value of the Katz two-variable p-adic L-function attached to a CM elliptic curve A at a point that lies *outside* the range of classical interpolation defining it. Rubin's formula expresses this special value in terms of the (square of) the p-adic formal group logarithm of a rational point on A. The reader should consult [9.5] or [9.6] for an exposition of Rubin's formula, and the first half of [BDP2] for an explanation of how this formula can be deduced from Theorem 9.2.

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Chapter 10

Stark-Heegner points attached to two-dimensional Artin representations

Beyond the fact that it leads to a formula for the p-adic logarithms of Heegner points in terms of special values of p-adic L-functions, one of the interests of the "p-adic Gross-Zagier theorem" formulated in Theorem 9.2 of the previous chapter is that it suggests a natural p-adic analytic construction of global points on elliptic curves in settings that go beyond what can be treated by the theory of complex multiplication. We will now describe this construction, which is much in the spirit of the constructions of units in number fields from leading terms of (p-adic) L-series, as described in the lectures of Samit Dasgupta and Matt Greenberg at this Winter School.

As in the previous chapter, let E be an elliptic curve over $\mathbb{Q}$ of conductor N and let $f \in S_2(\Gamma_0(N))$ be the associated normalised eigenform. Let

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}_2(\mathbb{C})$$

be a continuous, odd, irreducible two-dimensional Artin representation of $G_{\mathbb{Q}}$. Such a representation necessarily has finite image and therefore cuts out a finite extension K_ρ of $\mathbb{Q}$. Let $\tilde{\rho} : G_{\mathbb{Q}} \longrightarrow \mathbf{PGL}_2(\mathbb{C})$ be the projective representation obtained from ρ and let $\tilde{K}_\rho \subset K_\rho$ be the finite extension cut out by this representation. As is well-known (cf. §6 of [10.1]), the analysis of ρ can be divided into the following cases:

1. The representation $\tilde{\rho}$ has dihedral image and K_ρ is an abelian extension of an imaginary quadratic field K;

2. The representation $\tilde{\rho}$ has dihedral image and K_ρ is an abelian extension of a real quadratic field F. In that case, the representation ρ is necessarily induced from an abelian character of G_F which is even at one of the real places of F and odd at the other.

3. The representation $\tilde{\rho}$ has projective image isomorphic to A_4, S_4, or A_5.

For any such ρ, the Artin conjecture predicts the existence of a modular form

$$g = \sum_{n=1}^{\infty} a_n(g) q^n$$

of weight one, satisfying

$$\mathrm{Trace}(\rho(\mathrm{Frob}_p)) = a_p(g),$$

for all p at which ρ is unramified (and hence for which $\rho(\mathrm{Frob}_p))$ is a well-defined conjugacy class of elements in $\mathbf{GL}_2(\mathbb{C})$). The level of g is equal to the *Artin conductor* of ρ and its Nebentype character is the determinant of ρ, viewed as a Dirichlet character. This conjecture follows from the work of Langlands and Tunnel when ρ has solvable image, and was proved in a many icosahedral cases following a general program initiated by Taylor. (See [10.2] for an overview.) Nowadays the modularity of ρ is known in full generality as a consequence of the Serre Conjectures.

Let $(\tilde{\Lambda}_g, \underline{g})$ be the Hida family interpolating g in weight one, whose existence is guaranteed by Theorem 8.3, and consider the two-variable p-adic L-function $\mathscr{L}_p(\underline{g}, \underline{f})$.

When ρ is induced from an abelian character χ of an imaginary quadratic field, then g is a theta series of weight one attached to χ, and the Hida family $\underline{g}$ interpolating g in weight 1 can be chosen to be a family $\underline{\theta}$ of theta series attached to Hecke characters of K of varying infinity types. One recovers in this setting the p-adic L-function $\mathscr{L}_p(\underline{\theta}, \underline{f})$ that was the main object of Theorem 9.2. The following conjecture is a natural generalisation of Theorem 9.2 to the setting where g is not a CM form.

Conjecture 10.1. *Assume that the sign in the functional equation of $L(f \otimes g, s)$ is -1, so that $L(f \otimes g, 1) = 0$. Then there exists a global point $P_g^? \in E(K_\rho)$ and a global differential $\omega \in \Omega^1(E/\mathbb{Q})$ such that*

$$\mathscr{L}_p(\underline{g}, \underline{f})(\nu_{1,2,1}) = \log_\omega^2(P_g^?) \pmod{(K_f K_g)^\times}.$$

The point P_g is of infinite order if and only if $L'(f \otimes g, 1) \neq 0$.

In view of this, the following computational project arises naturally.

Exercise 10.2. *** Given the data of f and ρ as in this section, describe how the p-adic L-function $\mathscr{L}_p(\underline{g}, \underline{f})$ and its value at $\nu_{1,2,1}$ might be computed in practice. Can the Pollack-Stevens algorithm based on overconvergent modular symbols, which leads to a polynomial time algorithm for computing the Stark-Heegner points of [Dar01], be adapted to the setting of Hida's three variable p-adic Rankin L-series? Aside from its obvious theoretical interest, such an algorithm would be of great help in verifying Conjecture 10.1, and eventually in making it more precise.

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