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On computations of half-integral weight modular forms:

Thm (Serre-Stark): $M_{1/2}(\ldots) = \sum \text{span} < \sum_{\substack{r \equiv \mu (2m) \\ \mathbb{Z}}} q^{r^2/m} \mid \mu \bmod 2m >$

Now we focus on weight $3/2$. Fix $N \geq 1$, χ a Dirichlet character modulo $4N$ (χ is even).

$M_{3/2}(N, \chi) \ni F$ iff $Q = \frac{F}{\Theta^3}$ satisfies $Q(Az) = \chi(d)\, Q(z)$ for all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$ and F should be regular at the cusps. $\Theta = \sum_{n \in \mathbb{Z}} q^{n^2}$

The main theme is to describe the computation of the space

$S_{3/2}(N, \chi) =$ space of nontrivial cusp forms.

(trivial cusp forms: $\sum_{n \in \mathbb{Z}} \psi(n)\, n q^{tn^2}$ ψ DC mod ?, $t \in \mathbb{Z}_{\geq 1}$).

Example: $\frac{\eta(3z)\eta(12z)\eta(2\lambda z)^3}{\eta(6z)\eta(\lambda z)\eta(4\lambda z)} \Theta(z) \in S_{3/2}(3\lambda, (\frac{3\lambda}{\cdot}))$,

$\lambda \equiv 15 \pmod{24}$.

Thm: (Shimura, Niwa) For each $\square$-free $t \geq 1$ there is a Hecke-equivariant map

$$S_t : S_{3/2}(N, \chi) \longrightarrow S_2(2N, \chi^2).$$

Main property: (Waldspurger's thm): Let $f \in M_2(2N, \chi^2)$ be the Shimura lift of F, both Hecke eigenforms, then for every $\square$-free $t \geq 1$

$$\sqrt{t}\, \text{Cond}(N) \frac{L(1, f \otimes \chi_{\mathbb{Q}(\sqrt{-t})})}{\langle f, f \rangle} = \frac{|a_F(t)|^2}{\langle F, F \rangle}.$$

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$S_{3/2}(N,\chi)$: We look for a certain eigenform F in this space.

① Choose a modular form of wt $1/2$ (e.g. Θ)

② Compute a basis for $M_2(4N,\chi)$

③ Determine the image of $\cdot\,\Theta : S_{3/2}(N,\chi) \to M_2(4N,\chi)$.

④ Divide the form in the image by Θ.

⑤ Proceed with standard linear algebra to determine the Hecke eigenforms we are looking for.

§1 Main result:

$\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})] \supsetneq \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0 \circlearrowleft GL_2(\mathbb{Q})$

$c = \sum_{p\in\mathbb{P}^1(\mathbb{Q})} c(p)\, e_p \xrightarrow{\deg} \sum c(p)$

$G(N)_\ell = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det = \ell \ ,\ N\mid c,\ \gcd(a,N)=1 \right\}$

$\mathbb{C}(\bar{\chi}) = \Gamma(4N)$ - module : underlying space $\mathbb{C}$

$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A,\ z\right) \longmapsto \bar{\chi}(d)\, z$, $A \in \Gamma(4N)$

$\chi(a)$ — $G(N)_\ell$ - module.

'cuspidal modular symbols': $C(N,\chi) = \ker\left([\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0 \otimes \mathbb{C}(\bar{\chi})]_{\Gamma_0(4N)} \longrightarrow [\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})] \otimes \mathbb{C}(\bar{\chi})]_{\Gamma_0(4N)} \right)$

$E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ defines an involution on $C(N,\chi)$.

$C(N,\chi) = C(N,\chi)^+ \oplus C(N,\chi)^-$.

$$T(\ell)[c] = \sum_{M \in \Gamma(4N)\backslash G(N)_\ell} [Mc]$$

Recall: $C(N,\chi)^{\pm} \simeq S_2(N,\chi)^*$ $\quad \chi$ Dir. char. mod N.

Perfect pairing:

$$S_2(N,\chi) \times C(N,\chi)^{+} \longrightarrow \mathbb{C}$$
$$(f,[c]) \longmapsto \sum_{p \in \mathbb{P}^1(\mathbb{Q})} c(p) \left(\int_{p_0}^{p} f(z)\,dz + \int_{p_0}^{-p} f(z)\,dz \right)$$

with $p_0 \in \mathbb{P}^1(\mathbb{Q})$ fixed.

Thm: For $p,q \in \mathbb{P}^1(\mathbb{Q})$ and $\square$-free $t \geq 1$, set

$$f_{p,q,t} := \sum_{D \geq 0} a(D)\, e(D) \quad : \left(\text{for } D \neq A\square \right)$$

where $(4Nt = A\square$ and A $\square$-free$)$ and

$$a(D) = \sum_{\substack{Q \in F_N(4NDt) \\ = [a,b,c]}} \chi\left(\tfrac{a}{N}\right) \left(\frac{-4N/t}{a/N} \right) \frac{1}{2} \left(\operatorname{sign} Q(p) - \operatorname{sign} Q(q) \right).$$

and for $D = A\square$

$a(D)$ = same expression but with $ac \neq 0$ and correction terms.

the set

$$F_N(4NDt) = \{ [a,b,c] \text{ integral binary quad form} : b^2 - 4ac = 4NDt,\ N \mid a,\ 2N \mid b \}.$$

Then $f_{p,q;t} \in M_{3/2}(4N,\chi)$

- for each $\square$-free t there is a Hecke-equivariant map

$$L_t : C(2N,\chi^2) \longrightarrow S_{3/2}(4N,\chi)$$

s.t.

$$L_t\left(\left[\sum_{fin} c_{p,q} (e_p - e_q) \right] \right) \equiv \sum_{fin} c_{p,q} f_{p,q;t} \mod M_{3/2}^{triv}(4N,\chi)$$

Eis. series & triv. cusp forms

- $\displaystyle \sum_{\substack{t \geq 1 \\ \square\text{-free}}} \operatorname{Im} L_t = S_{3/2}(4N,\chi).$

Idea of proof:

$$S_{3/2}(4N,\chi) \xrightarrow[\langle\, ,\, 2\vartheta_{(\cdot,z)}\rangle]{S_t} M_2(2N,\chi) \xrightarrow[\mathrm{per}_+]{} \left(C(2N,\chi)^+\right)^*$$

dualize

$$L_t^* : C(2N,\chi) \longrightarrow S_{3/2}(4N,\chi)^* \underset{\langle\, ,\, \rangle}{\simeq} S_{3/2}\left(4N,\chi\left(\tfrac{4N}{\cdot}\right)\right).$$

Then

$$\sum L_t^* = S_{3/2}(\cdot).$$

$$L_t^*([c]) \equiv \int_{c^+} \vartheta(\tau,z)\,d\tau \quad (\mathrm{mod}\ M_{3/2}^{\mathrm{triv}})$$