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Hida families and p-adic triple product L-functions

§1 Introduction:

Let $f_i \in S_{k_i}(\Gamma_0(N_i), \chi_i)$ $i=1,2,3$. We can attach

$L(s, f_1 \otimes f_2 \otimes f_3)$ L-function attached to $\rho_{f_1} \otimes \rho_{f_2} \otimes \rho_{f_3}$.

$L(s, f_1 \otimes f_2 \otimes f_3) = \prod_\ell P_\ell(s)^{-1}$ s.t. for $\ell \nmid N_1N_2N_3$ we

have $P_\ell(s) = \det\left(1_8 - \begin{pmatrix} \alpha_{f_1}(\ell) & \\ & \beta_{f_1}(\ell)\end{pmatrix} \otimes \begin{pmatrix} \alpha_{f_2}(\ell) & \\ & \beta_{f_2}(\ell)\end{pmatrix} \otimes \begin{pmatrix} \alpha_{f_3}(\ell) & \\ & \beta_{f_3}(\ell)\end{pmatrix} \ell^{-s}\right)$

where $\alpha_i(\ell), \beta_i(\ell)$ roots of $X^2 - a_\ell(f_i)X + \chi_i(\ell)\ell^{k_i-1}$,

$a_\ell(f_i) = \ell^{th}$ F.c. of f_i.

$L(s, f_1 \otimes f_2 \otimes f_3)$ converges for $\operatorname{Re}(s) > \frac{k_1+k_2+k_3-1}{2}$.

Assume $\chi_1\chi_2\chi_3 = 1$.

Garrett, Piatetski-Shapiro, Rallis ...

· Analytic continuation and functional equation

$$L(s, f_1 \otimes f_2 \otimes f_3) \longleftrightarrow L(w+1-s, f_1 \otimes f_2 \otimes f_3)$$
$$w = k_1+k_2+k_3-3.$$

We are interested in the p-adic behavior for the central value

$L(\frac{w+1}{2}, f_1 \otimes f_2 \otimes f_3)$ when f_1, f_2, f_3 vary in Hida families.

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§2 Algebraicity and periods:

Define $\Sigma_i = \left\{ \begin{array}{l} \underline{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3_{\geq 2} \\ k_1 + k_2 + k_3 \equiv 0 \pmod 2 \end{array} : 2k_i \geq k_1 + k_2 + k_3 \right\}$ $i = 1, 2, 3.$

$\Sigma_{bal} = \left\{ \text{"} \quad \text{..} \quad : k_1 \underset{k_3}{\triangle} k_2 \right\}$

$\left\{ \underline{k} \in \mathbb{Z}^3_{\geq 2},\ k_1 + k_2 + k_3 \equiv 0 \pmod 2 \right\} = \Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3 \sqcup \Sigma_{bal}.$

If $\underline{k} \in \Sigma_i$, $\Omega_{f_1 \otimes f_2 \otimes f_3} := \Omega^2_{f_i}$, $\Omega_{f_i} := \dfrac{\|f_i\|^2_{\Gamma_0(N_i)}}{\eta_{f_i}}$

(unbalanced case) $\qquad \uparrow$ congruence #.

If $\underline{k} \in \Sigma_{bal}$, $\Omega_{f_1 \otimes f_2 \otimes f_3} = \Omega_{f_1} \Omega_{f_2} \Omega_{f_3}$

(balanced case)

- $\dfrac{L\left(\frac{w+1}{2}, f_1 \otimes f_2 \otimes f_3\right)}{\Omega_{f_1 \otimes f_2 \otimes f_3}} \in \mathbb{Q}(f_1, f_2, f_3).$

  - Harris-Kudla in unbalanced case.
  - Garrett, Orloff, ... in balanced case.

§3 p-adic L-functions in the balanced case:

$\Gamma = 1 + p\mathbb{Z}_p$ (p prime, $p > 2$) $\quad \mathcal{O} = \mathcal{O}_{\mathbb{Q}(f_1, f_2, f_3), p}$

$(\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p).$

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Let $\Lambda = \mathcal{O}[\![\Gamma]\!] \simeq \mathcal{O}[\![T]\!]$. Let f, g, h be three Hida families of tame level N_1, N_2, N_3 with trivial tame character. (for simplicity assume $f, g, h \in \Lambda[\![q]\!]$).

$$\mathcal{X} = \{ x : \Lambda \to \overline{\mathbb{Q}}_p : x(T) = \zeta_x (1+p)^{k_x} - 1 ,\ \zeta_x \in \mu_{p^\infty},\ k_x \in \mathbb{Z}_{\geq 2} \}.$$

$$x \in \mathcal{X},\quad \epsilon_x : \mathbb{A}_{\mathbb{Q}}^\times \to \overline{\mathbb{Q}}^\times,\quad \epsilon_x(u) = \zeta_x^{\epsilon_{cycl}^{-1}(\mathrm{rec}(u)|_{\mathbb{Q}_\infty})}.$$

rec = reciprocity law

$\mathbb{Q}_\infty/\mathbb{Q}$ = cyclo. $\mathbb{Z}_p$-ext.

ϵ_{cyc}. p-adic cyclo. char.

$$\mathcal{X}_{bal} = \{ (x,y,z) \in \mathcal{X}^3 : (k_x, k_y, k_z) \in \Sigma_{bal} \}.$$

<u>Thm</u>: Assume: (1) $M = \gcd(N_1, N_2, N_3)$ is square-free and N_1/M, N_2/M, N_3/M coprime to each other.

(2) $\#\{ \ell \mid M : a_\ell(f) a_\ell(g) a_\ell(h) = -1 \}$ is odd.

(3) (CR) (1) $\overline{\rho}_{f,p}$ is abs. irred.

(2) $\overline{\rho}_f|_{I_\ell}$ is ramified for $\ell \mid M$ and $\ell^2 \equiv 1 \pmod p$

$\exists !\ \mathcal{L}_p \in \mathcal{O}[\![\Gamma\times\Gamma\times\Gamma]\!] \simeq \mathcal{O}[\![T_1, T_2, T_3]\!]$ s.t. $\forall\ \theta = (x,y,z) \in \mathcal{X}_{bal}$,

$$\mathcal{L}_p(\theta) = \frac{L\left(\frac{w_\theta+1}{2}, f_x \otimes g_y \otimes h_z \otimes \omega^{\frac{k_x+k_y+k_z}{2}} \sqrt{\epsilon_x \epsilon_y \epsilon_z}\right)}{\Omega_{f_x \otimes g_y \otimes h_z}}$$

$$\cdot\ \Gamma_{\mathbb{C}}\left(\frac{w_\theta+1}{2}\right) \Gamma_{\mathbb{C}}\left(\frac{k_x+k_y-k_z}{2}\right) \Gamma_{\mathbb{C}}\left(\frac{k_x+k_z-k_y}{2}\right) \Gamma_{\mathbb{C}}\left(\frac{k_y+k_z-k_x}{2}\right)$$

$\cdot\ \mathcal{E}_{f_x \otimes g_y \otimes h_z}$, where

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- $w_\Theta = k_x + k_y + k_z - 3$

- $\Gamma_{\mathbb{C}}(s) = 2\cdot(2\pi)^{-s}\Gamma(s)$

- $\mathcal{E}_{f_x \otimes g_y \otimes h_z}$ modified Euler factor at p

Definition of $\mathcal{E}_{f_x \otimes g_y \otimes h_z}$

$$V_x = WD(\rho_{f_x,\ell}|_{G_{\mathbb{Q}_p}})^{F\text{-s.s}} \quad : \text{the Weil-Deligne rep. of } W_{\mathbb{Q}_p}.$$

$\dim V_x = 2$

$F^+V_x \subset V_x$, $\dim_{\mathbb{C}} F^+V_x = 1$ unramified, Frob_p acts by p-adic unit.

$$V_\Theta = V_x \otimes V_y \otimes V_z.$$

$$\cup|$$

$$U_\Theta = F^+V_x \otimes F^+V_y \otimes V_z + V_x \otimes F^+V_y \otimes F^+V_z + F^+V_x \otimes V_y \otimes F^+V_z$$

$\dim_{\mathbb{C}} U_\Theta = 4$ sub $W_{\mathbb{Q}_p}$-module.

Then

$$\mathcal{E}_{f_x \otimes g_y \otimes h_z} := \mathcal{E}\left(U_\Theta, \frac{w_\Theta+1}{2}\right) L\left(V_\Theta/U_\Theta, \frac{w_\Theta+1}{2}\right)^{-2}$$

<u>Remark:</u> In the unbalanced case, there are works of Harris-Tilouine and Darmon-Rotger.

§4 Sketch of the construction of $\mathcal{L}_p$

D definite quaternion algebra ramified at ∞N^- where

$N^- = \prod_\ell \ell$: $a_\ell(f)\,a_\ell(g)\,a_\ell(h) = -1$.

$\hat{D} = D \otimes_{\mathbb{Q}} \hat{\mathbb{Q}}$ R : Eichler order of level $N = \mathrm{lcm}\{\frac{N_1}{M}, \frac{N_2}{M}, \frac{N_3}{M}\}$.

Fix $D \otimes \mathbb{Q}_p \cong \mathrm{Mat}_2(\mathbb{Q}_p)$.

$$X = D^\times \backslash \hat{D}^\times / \hat{R}^{(p)\times} \circlearrowleft GL_2(\mathbb{Q}_p)$$

$$U_1(p^n) = \left\{ g \in GL_2(\mathbb{Z}_p) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{p^n} \right\}$$

$X_1(p^n) = X / U_1(p^n)$; $J_n = \mathcal{O}[X_1(p^n)]$.

On J_n, we have action of $\Gamma = 1 + p\mathbb{Z}_p$

$a \in 1 + p\mathbb{Z}_p$, $\langle a \rangle [x] = \left[x \begin{pmatrix} 1 & \\ & a^{-1} \end{pmatrix} \right]$.

U_p, T_p are usual Hecke operators acting on J_n.

$J_{n+1} \twoheadrightarrow J_n$.

$J_\infty = \varinjlim J_n \circlearrowleft$ Λ-module

$J_\infty^{\mathrm{ord}} = e J_\infty$ $e = \varinjlim U_p^{n!}$

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$$S^{\mathrm{ord}}(N,\Lambda) = \mathrm{Hom}_\Lambda(J_\infty^{\mathrm{ord}}, \Lambda)$$

<u>Thm (Hida):</u> $\exists\ \mathcal{L}_D(g_D, h_D) \in S^{\mathrm{ord}}(N,\Lambda)$ s.t.

$$T_\ell \mathcal{L}_D = a_\ell(\mathcal{L})\mathcal{L}_D \quad \forall\, \ell \nmid N_i p.$$

<u>Prop.</u> Under the assumptions (CR), we can choose $f_D, (g_D, h_D)$ s.t.

$$f_D \not\equiv 0 \ (\mathrm{mod}\ m_\Lambda).$$

<u>Regularized diagonal cycles:</u>

$$\Delta_n := \sum_{[x]\in X_1(p^n)} \sum_{\substack{b_1\in(\mathbb{Z}/p^n\mathbb{Z})^\times \\ b_2\in\mathbb{Z}/p^n\mathbb{Z}}} \left[x\begin{pmatrix} p^n & b_1+b_2 \\ 0 & b_1^{-1}\end{pmatrix}, x\begin{pmatrix} p^n & b_2 \\ 0 & b_1^{-1}\end{pmatrix}, x\begin{pmatrix} 0 & b_1 \\ -p^n & b_2\end{pmatrix}\right]$$

$$\in J_n^{\mathrm{ord}} \otimes J_n^{\mathrm{ord}} \otimes J_n^{\mathrm{ord}}$$

$$\Delta_n^+ := U_p^{-n} \otimes U_p^{-n} \otimes U_p^{-n}(\Delta_n)$$

<u>Lemma:</u> $\Delta_{n+1}^+ \equiv \Delta_n^+$ in $(J_n^{\mathrm{ord}})^{\otimes 3}$ $(J_{n+1}^{\mathrm{ord}} \twoheadrightarrow J_n^{\mathrm{ord}})$

$$\leadsto \quad \Delta_\infty^+ = \varprojlim_n \Delta_n^+$$

$$\mathcal{L}_D^+ = \mathcal{L}_D \otimes \langle\ \rangle_T^{1/2}$$

$$\langle\cdot\rangle_T : \hat{D}^\times \xrightarrow{v} \hat{\mathbb{Q}}^\times \longrightarrow \Lambda^\times$$

$$a \longmapsto (1+T)^{\frac{\log \varepsilon_{cyc}(a)}{\log(1+p)}}$$

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$$\Theta_\infty := \mathcal{L}_D^+ \otimes g_D^+ \otimes h_D^+ (\Delta_\infty^+) \in \mathcal{O}[[T_1, T_2, T_3]].$$

Define $\mathcal{L}_p := \Theta_\infty^2$.

The proof of the theorem boils down to

(1) the interpretation of $\Theta_\infty(\theta)$ as a global trilinear period integral

$$\int_{\mathbb{A}^\times D^\times \backslash D^\times_{\mathbb{A}}} f_{D_x}^{-1} \otimes g_{D_y}^{-1} \otimes h_{D_z}^{-1}(x)\,dx.$$

(2) Ichino's formula

$$\Theta_\infty^2(\theta) = L\text{-value} \prod_\ell \underset{\text{local integrals.}}{I_\ell}$$