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8-15-16

PSI

Moduli stacks of potentially Barsotti-Tate Galois representations

Joint work w/ Caraiani, Emerton, Gee.

Problem: Construct moduli of p-adic Galois representations ρ (with some specified p-adic Hodge theoretic properties) in which $\bar{\rho}$ is allowed to vary, and moduli of such $\bar{\rho}$.

Motivating example: For $x \in \bar{\mathbb{Z}}_p$, define $M_x = \bar{\mathbb{Z}}_p e_1 \oplus \bar{\mathbb{Z}}_p e_2$.

$\phi: M_x \to M_x$ given by $\begin{pmatrix} 0 & 1 \\ -p & x \end{pmatrix}$.

$$\mathrm{Fil}^0 M_x = M_x, \quad \mathrm{Fil}^1 M_x = \bar{\mathbb{Z}}_p e_1, \quad \mathrm{Fil}^2 M_x = 0.$$

Then M_x is a Fontaine-Laffaille module

$\overset{FL}{\Longleftrightarrow}$ $\rho_x: G_{\mathbb{Q}_p} \to GL_2(\bar{\mathbb{Z}}_p)$ that is Barsotti-Tate (crystalline w/ HT wts 0,1).

Consider reductions mod p

- $\bar{\rho}_x \cong \begin{pmatrix} \mathrm{un}(\bar{x}^{-1})\bar{\varepsilon} & * \\ 0 & \mathrm{un}(\bar{x}) \end{pmatrix}$ if $\bar{x} \neq 0$.

(if $\bar{x} = \pm 1$, then $*$ is peu-ramifié

- $\bar{\rho}_x$ is irred. if $\bar{x} = 0$.

$\{\bar{\rho}_x\}$ is generically reducible, but specializes to an irred. point.

Notation:

$p > 2$, $K/\mathbb{Q}_p$ finite ext of inertial degree f.

K'/K tame Galois ext., $\pi' \in K'$ a unif. chosen so $(\pi')^{e(K'/K)} = \pi \in K$, k' = res. field of K'.

$E(u)$ = Eisenstein poly. for π' over $W(k')$.

· Let $\mathfrak{S} = W(k')[[u]]$, $\varphi(\sum a_i u^i) = \sum \varphi(a_i) u^{p^i}$

$\circlearrowleft$ $\mathrm{Gal}(K'/K)$ $\qquad g(a_i u^i) = g(a_i) \frac{g\pi'}{\pi'} u^i$.

def $A = \mathbb{Z}_p$-alg., $\mathfrak{S}_A = \mathfrak{S} \hat{\otimes}_{\mathbb{Z}_p} A$, extend φ, Gal-action to $\mathfrak{S}_A$, $\mathfrak{S}_A[\frac{1}{u}]$.

Def: An étale φ-module (with descent data and A-coeffs) is a triple $(M, \phi, \{\hat{g}\}_{g \in \mathrm{Gal}(K'/K)})$

· M is fin. gen. $\mathfrak{S}_A[\frac{1}{u}]$-module

· $\phi: M \to M$ is a φ-semilinear map s.t. $\varphi^* = 1 \otimes \varphi : \varphi^* M = \mathfrak{S}_A[\frac{1}{u}] \otimes_{\varphi} M \xrightarrow{\sim} M$.

· $\hat{g}: M \to M$ commutes with φ, $\hat{g}\hat{h} = \widehat{gh}$, $\hat{1} = id$.
Galois-semilinear.

Def: Let $h \geq 1$. A (Breuil)-Kisin module of height h is a triple $(\mathfrak{M}, \phi, \{\hat{g}\})$ s.t.

· $\mathfrak{M}$ is a fin. gen. $\mathfrak{S}_A$-module

· $\phi: \mathfrak{M} \to \mathfrak{M}$ is φ-semilinear s.t. $\mathfrak{M}[\frac{1}{u}]$ is an étale φ-module.

· $\hat{g}: \mathfrak{M} \to \mathfrak{M}$ · same conditions.

· cokernel of $\varphi^*: \varphi^* \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^h$.

Def: An étale φ-mod. (resp. Kisin module) is locally free if it is projective & fpqc locally free on Spec A.

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<u>Connections with Galois representations:</u>

Fix a system π_n of $(p^n)^{th}$ roots of π.
Let $K_\infty = K(\pi_n, n \geq 1)$.

<u>Fontaine:</u> If A is a local $\mathbb{Z}_p$-algebra, $|A| < \infty$, then there exists an equivalence

$$\left\{ \begin{array}{l} \text{étale } \varphi\text{-mods} \\ \text{w/ } A \text{ coeffs} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{cts. reps. of } G_{K_\infty} \\ \text{on fin. } A\text{-modules} \end{array} \right\}$$

Say a G_{K_∞}-rep. has $ht \leq h$ if it $\leftrightarrow M = \mathfrak{M}[\frac{1}{u}]$ $ht \leq h$.

<u>Kisin:</u>

$$\left\{ \begin{array}{l} \text{Kisin modules with} \\ \mathbb{Z}/p^a\mathbb{Z}\text{-coefficients } ht \leq 1 \\ \text{\& descent data} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{fin. flat } \mathbb{Z}/p^a\mathbb{Z}\text{-mod.} \\ \text{schemes}/\mathcal{O}_{K'} \text{ \& } \ldots \text{ descent data} \end{array} \right\}$$

generic fiber

$$\left\{ \begin{array}{l} \text{pot. fin. flat reps of} \\ G_K \text{ with } \mathbb{Z}/p^a\mathbb{Z}\text{-coeffs} \\ \text{that become flat over } K' \end{array} \right\}$$

<u>Df:</u> Fix d, fix $a, h \geq 1$. Define étale stacks over $\mathbb{Z}/p^a\mathbb{Z}$: for any $\mathbb{Z}/p^a\mathbb{Z}$-algebra A,

$$\cdot\ \mathcal{C}_a^{\leq h, dd}(A) := \left\{ \begin{array}{l} \text{locally free}_{\text{rank } d} \text{ Kisin modules of } ht \leq h, dd \\ \text{\& } A\text{-coeffs.} \end{array} \right\}$$

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- $\mathcal{R}_a^{dd}(A) = \left\{ \begin{array}{l} \text{loc. free étale } \varphi\text{-mods with } A\text{-} \\ \text{coeffs. \& descent data} \end{array} \right\}$

<u>Thm (Pappas - Rapoport):</u>

- $\mathcal{C}_a^{\le h, dd}$ is an algebraic stack of finite type over $\mathbb{Z}/p^a\mathbb{Z}$.

- $\mathcal{C}_a^{\le h, dd} \to \mathcal{R}_a^{dd}$ is proper, fin. presentation, and representable by algebraic spaces.

$\mathfrak{M} \mapsto \mathfrak{M}[\frac{1}{u}]$

<u>Thm (Emerton - Gee):</u> The map $\mathcal{C}_a^{\le h\, dd} \longrightarrow \mathcal{R}_a^{dd}$ has a "scheme-theoretic image" $\mathcal{R}_a^{\le h, dd}$ which is an algebraic stack of finite type. and

$$\mathcal{C}_a^{\le h, dd} \longrightarrow \mathcal{R}_a^{\le h, dd} \text{ is proper.}$$

<u>Cor:</u> $\mathcal{R}_a = \varinjlim_h \mathcal{R}_a^{\le h, dd}$ is ind. algebraic.

Think of $\mathcal{R}_a^{\le h, dd}$ as G_{K_∞}-representations of ht $\le h$.

and $\mathcal{R}_a^{\le 1, dd}$ as a module of G_K- reps. fin. flat over $G_{K'}$.

For the rest of the talk take $d = 2$, $h = 1$. [Replace ≤ 1 by BT]

Impose "Kottwitz condition" $\Rightarrow$ HT wts are $\{0, 1\}$.

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Further refinement: descent data acts through a pair of characters χ_1, χ_2 of $I(K'/K)$, $\tau = \chi_1 \oplus \chi_2$.

$\Rightarrow \mathcal{C}_a^{BT,dd} \supset \mathcal{C}_a^{BT,\tau}$.

$\mathcal{C}_a^{BT,dd}$ with scheme theoretic image Z_a in $\mathcal{R}_a^{dd}$

$\mathcal{C}_a^{BT,\tau}$ with scheme theoretic image Z_a^{τ} in Z_a

Write $\sigma(\tau)$ = irred. $\bar{\mathbb{Q}}_p$-rep. of $GL_2(k)$ corresp. to τ under inertial LL.

<u>Thm (CEGS):</u>

(1) $\mathcal{C}_1^{BT,dd}$, $\mathcal{C}_1^{BT,\tau}$, Z_1, Z_1^{τ} are all reduced and equidimensional of dim. $[K:\mathbb{Q}_p]$.

(2) $\mathcal{C}_1^{BT,\tau}$ has 2^f components indexed by $F=0$ or $V=0$ on i^{th} piece of the Dieud module group scheme.

(3) closed components of $Z_1 \leftrightarrow$ non-Steinberg Serre weights. $\bar{\rho}$ lies in $Z(\sigma) \Leftrightarrow \bar{\rho}$ admits σ as a Serre weight.

(4) closed components of $Z_1^{\tau} \leftrightarrow Z(\sigma)$ for $\sigma \in JH(\overline{\sigma(\tau)})$.
$\bar{\rho}$ lies in $\leftrightarrow$ $\bar{\rho}$ with Serre weight in

(5) $\exists$ open dense $U \subseteq Z_1^{\tau}$ s.t. $(\mathcal{C}_1^{\tau})_U \xrightarrow{\sim} U$.