Critical values of L-functions:

X = proj. smth variety $/\mathbb{Q}$

One has an associated L-function $L(X,s)$, the Hasse-Weil L-function.

$$L(X,s) = \prod_p L_p(X \bmod p, s).$$

$$L_p(X \bmod p, s) = \exp\left(-\sum_{n\geq 1} (\# X \bmod p)(\mathbb{F}_{p^n}) \frac{T^n}{n}\right)\Bigg|_{T=p^{-s}}. \quad \left(\text{at least for primes of good reduction}\right)$$

Weil conjectured properties of $L_p(X \bmod p, s)$

Weil conjectured that there should be a cohomology theory to explain $L_p(X \bmod p, s)$.

For example, let X = elliptic curve. $= E$

$$L_p(E \bmod p, s) = \left(\frac{(1-p^{-s})(1-p^{1-s})}{(1-a_p p^{-s}+p^{1-2s})}\right)^{-1} ?$$

$$= \frac{\zeta(s)\,\zeta(s-1)}{L(H^1(E),s)}$$

where $L(H^1(E), s) = (1 - a_p p^{-s} + p^{1-2s})^{-1}$. We want to study meromorphic continuation, so we want to study $L(H^1(E),s)$. This should be, if there is a cohomology theory as Weil believed

$$\frac{L(H^0(E),s)\, L(H^2(E),s)}{L(H^1(E),s)}.$$

In general, one would want to have

$$L(X,s) = \prod_{i=0}^{2d} L(H^i(X),s)^{(-1)^i}.$$

So the L-function should be defined as a product of cohomologically defined L-functions. This is true as was proved by Grothendieck and his school.

There is even a further decomposition that is typical.

Consider $X_0(N)$ a modular curve.

$$L(X_0(N), s) = \frac{L(H^0(X_0(N)), s)\, L(H^2(X_0(N)), s)}{\prod_{\substack{f \text{ Hecke} \\ \text{e.f.}}} L(f,s)}$$

(we aren't addressing primes dividing the level...)

so $$L(H^1(X_0(N)), s) = \prod_{\substack{f \text{ Hecke} \\ \text{e.f.}}} L(f,s).$$

This leads one to seek a theory that has one irreducible object for each "irreducible" L-function.

Theory of Motives:

One can think of this in analogy to integrals. There are lots of kinds, Lebesgue, Riemann, etc.. you generally know which from the context.

Similarly, there are many types of motives:

- Grothendieck motives ← we are working with these.
- motives for absolute Hodge cycles
- chow motives
- Voevodsky motives
- l-adic representations
- motives as realizations
- Levine motives.

⋮

So $L(M,s)$ is defined for our motives.

Two types of questions:

① analytic number theory question: what info about the sets of data used to define L_p's can knowing the analytic data about the

③ Borel's Thm

$$L^*(\mathrm{Spec}(K), 1-n) = q\, R(K_{2n-1}(\mathcal{O}_K)) \qquad q \in \mathbb{Q}^\times.$$

$n \geq 2$

$$t_{\mathrm{Spec}(K), 1-n} = \mathrm{rk}\, K_{2n-1}(\mathcal{O}_K)$$

④ Birch-Swinnerton Dyer conj. (1960's)

$$L(H^1(E), s) =$$

$$L^*(H^1(E), 1) = \left(P_E^{\leftarrow \text{period}}\, |Ш|\, |\det\langle p_i, p_j\rangle| \,/\, |E(\mathbb{Q})_{tors}|^2\right) \cdot \prod_{p|\Delta} c_p$$

$$t_{H^1(E),1} = \mathrm{rk}\,(E(\mathbb{Q}))$$

ω_{can} Neron differential.

$$P_E = \left|\int_{E(\mathbb{R})} \omega_{can}\right|.$$

We are going to look at this up to rational numbers, so eliminate the finite groups. Seek in the form:

$\left\{\begin{array}{l} L^*(M,n) = q\,(\text{something interesting})\ P_M\ \text{ht's regulators.} \\ t_{M,n} \end{array}\right.$

type of conjecture Beilinson made in the early 1980's.

We can further simplify by eliminating the rank. Then either $L(M,n)=0$ or $L(M,n) = q \cdot P_M^{\leftarrow \text{period}}$.

$L(M,s) = * L(M^\vee, 1-s)$ functional equation.

M has a weight w.

M_B = topological cohomology.

$M_B \otimes \mathbb{C} = \bigoplus M^{p,q}$, $\overline{M^{p,q}} = M^{q,p}$.

M irreducible, $M^{p,q} = \{0\}$ unless $p+q = w$ weight.

The functional equation really $L(M,s) = x\, L(M, w+1-s)$.

Expect $L(M,s)$ absolutely convergent for $\mathrm{Re}(s) > \frac{w}{2}+1$,
$L(M,n) \neq 0$ if $n \geq \frac{w}{2}+1$ and non-zero at $\frac{w}{2}+1$.
Evidently, knowing $L^*(M,n)$ for $n \leq \frac{w+1}{2}$ determines it for
$\frac{w+1}{2} - n \geq \frac{w+1}{2}$.

The rank part of the statement applies only to $n \leq \frac{w+1}{2}$.

Deligne (1977): Conjectured the P_M that should work.

$$L^*(M,n) = q\, P_M \quad , q \in \mathbb{Q} \quad \text{if } n \text{ is critical.}$$

n critical ~~[illegible]~~ ① $n = \frac{w+1}{2}$ (w odd)

② $t_{M,n}$ ~~[illegible]~~ is zero. if philosophy holds.

i.e., critical iff $n = \frac{w+1}{2}$ or all the f.g groups have 0 rank.

$n = \frac{w+1}{2}$

~~$t_{M,\frac{w+1}{2}}$ = mystery, ∃ w-fold, M made using w-fold X.~~

~~Conjecture:~~

Conjecture: ∃ w-fold X s.t. M is made using the w-fold

$t_{M,\frac{w+1}{2}}$ = mystery = on X have $CH^{\frac{w+1}{2}} = \dfrac{\text{homologically} \sim 0 \text{ codim } \frac{w+1}{2} \text{ cycles}}{(\text{rational equiv.})}$

and

$$L^*(H^w(X), \tfrac{w+1}{2}) = q \cdot P_X \, |\det \langle p_i, p_j \rangle| \qquad (\text{general conj.}).$$

(Bloch - Beilinson conj).

What about $n < \frac{w+1}{2}$? $\quad t_{M,n} = ?$

Functional equation:

$$L_\infty(M,s)\, L(M,s) = L_\infty(M, w+1-s)\, L(M, w+1-s).$$

$w+1-n > \frac{w+1}{2}$, so

$$L_\infty(M,n)\, L(M,n) = L_\infty(M, w+1-n)\, \underbrace{L(M, w+1-n)}_{\neq 0}.$$

So $\operatorname{ord}_{s=n} L(M,s) = \operatorname{ord}_{s=w+1-n} L_\infty(M, w+1-n)$ ← $=0$. turns out..

$- \operatorname{ord}_{s=n} L_\infty(M,s)$

So $\operatorname{ord}_{s=n} L(M,s) = - \operatorname{ord}_{s=n} L_\infty(M,s)$

This says rk A_n is determined by data coming from Hodge theory.

<u>Example:</u> Look at the ζ-function.

$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) = \cdots$

~~Because exception~~ $\Gamma(\frac{s}{2})$ has poles at $0,-2,-4,\ldots$

$$\operatorname{rk}\left(K_{2|n|+1}(\mathbb{Z})\right) = \begin{cases} 0 & n \in 2\mathbb{Z}+1 \\ 1 & n \in 2\mathbb{Z} \end{cases}$$

<u>Serre's rules for Γ-factors $(L_\infty(M,s))$:</u>

$F_\infty \curvearrowright M_B \otimes \mathbb{C} = \bigoplus_{p<q} (M_B^{p,q} \oplus M_B^{q,p}) \oplus M^{\frac{w}{2},\frac{w}{2},+} \oplus M^{\frac{w}{2},\frac{w}{2},-}$.

F_∞ = complex conj.; $F_\infty = +1$, $F_\infty = -1$.

$\dim M^{p,q} = d^{p,q}$ and $d^{\frac{w}{2},\frac{w}{2},\pm}$.

$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\,\Gamma(s)$

$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\,\Gamma(\frac{s}{2})$

$$L_\infty(M,s) = \prod_{p<\frac{w}{2}} \Gamma_{\mathbb{C}}(s-p)^{d^{p,w-p}}\;\Gamma_{\mathbb{R}}(s-\tfrac{w}{2})^{d^{\frac{w}{2},\frac{w}{2},(-1)^{\frac{w}{2}}}}\;\Gamma_{\mathbb{R}}(s+1-\tfrac{w}{2})^{d^{\frac{w}{2},\frac{w}{2},(-1)^{\frac{w}{2}+1}}}$$

$\Gamma(s)$ is entire except for simple poles at $0,-1,-2,\ldots$

What are the critical integers?

① Assume no $(\frac{w}{2},\frac{w}{2})$ classes

Let $p<q$ be maximal s.t. $M^{p,q} \neq \{0\}$. Then $[p+1,q]$ is

the set of critical integers.

<u>Example</u>: $f \in S_k(\Gamma)$ newform.

$$M_{f,B} = M^{k-1,0} \oplus M^{0,k-1} \quad \text{so critical values}$$

are $\{1, \ldots, k-1\}$.

If $M^{w/2,w/2} \neq 0$ for both + and − if both $\Gamma_{\mathbb{R}}(s-p)$ and $\Gamma_{\mathbb{R}}(s+1-p)$ show up, then the critical strip is killed.

<u>Example</u>: $f \in S_k$, $g \in S_\ell$ then

$$(M_f \otimes M_g)_B \otimes \mathbb{C} = M^{k+\ell-2,0} \oplus M^{0,k+\ell-2} \oplus M^{k-1,\ell-1} \oplus M^{\ell-1,k-1}.$$

If $k > \ell$, $\{\ell, k-1\}$ = critical strip.

Shimura proved a special value result here as well..

<u>Example</u>: f, g, h (wt k, ℓ, m) and triple product.

Formula proved by Garrett, Orloff, Garrett & Harris, for critical integers provided $k+\ell > m$. For $k+\ell < m$, proved by Harris and Kudla in the course of settling a conjecture of Jacquet.

Deligne considered only the case $n=0$. Why? Because $L(M(n),s) = L(M,s+n)$, so one can Tate twist to other values.

Deligne conjectures if 0 is critical for M, either $L(M,0)=0$ or $L(M,0) = q\, \underset{\uparrow \text{period}}{c^+(M)}$ and $q \in \mathbb{Q}^\times$.

One has a map

$$\underset{\substack{\uparrow \\ \text{top. cohom}}}{M_B \otimes \mathbb{C}} \xrightarrow{I_\infty} \underset{\substack{\uparrow \\ \text{de Rham}}}{M_{DR} \otimes \mathbb{C}}$$

M_{DR} has Hodge filtration.

$$F^i M_{DR} = \left(I_\infty^{*}\left(\bigoplus_{p \geq j} M^{p,q}\right)\right) \cap M_{DR}.$$

<u>Prop:</u> If M is critical at 0, then

$$\dim\left(M_{DR}/F^0 M_{DR}\right) = \dim M_B^+ \qquad +1 \text{ eigenspace } \cancel{\text{e.s.}} \text{ of c.c.}$$

and

$$\begin{tikzcd} I_\infty^+ : M_B^+ \otimes \mathbb{C} \arrow[r] \arrow[rr, bend right, "\simeq"'] & M_{DR} \otimes \mathbb{C} \arrow[r] & \left(M_{DR}/F^0 M_{DR}\right) \otimes \mathbb{C}. \end{tikzcd}$$

As Deligne says: Let $\gamma_1, \ldots, \gamma_{d^+}$ be a basis of M_B^+. Let $w_1, \ldots, w_{d^+}$ be a basis of $M_{DR}/F^0 M_{DR}$. Let

$I_\infty^+(\gamma_1, \ldots, \gamma_{d^+}, w_1, \ldots, w_{d^+}) = d^+ \times d^+$ matrix of I_{d^+} in these..

Define ~~I~~ $c^+(M) = \det I_\infty^+(\gamma_1, \ldots, \gamma_{d^+}, w_1, \ldots, w_{d^+})$

Then

$$L(M,0) = \begin{cases} 0 \\ q\, c^+(M \end{cases}$$

This is known for modular forms, Hecke L-series, and many other things.