ON THE EISENSTEIN IDEAL FOR $U(2,1)$

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This is a report on work in progress. We want to outline a possible proof of the Iwasawa conjecture for CM fields. The strategy is to use congruences between Eisenstein series and cusp forms on $U(2,1)$ and can be seen as a direct generalization of Wiles' proof of Iwasawa conjecture for totally real fields. We will actually see that there are still some technical obstructions, thus leaving the conjecture unproven so far.

Notation

Let K be a CM field, i.e. a totally imaginary quadratic extension of a totally real field F of degree d, O_K (resp. O_F) the ring of integers of K (resp. of F). We denote by $\tau_{K/F}$ the quadratic Hecke character of F associated to K/F and $\Delta_{K/F}$ the discriminant of K/F. The group of pth roots of unity in $\bar{\mathbf{Q}}$ will be denoted by μ_p.

We fix an odd rational prime p and we suppose:

(ord): *every prime of F above p splits in K*

Let c be the non-trivial element of $Gal(K/F)$. For any set X on which $Gal(K/F)$ acts, we write $X^{\pm}$ for the set $\{x \in X; x^c = x^{\pm}\}$. We fix two embeddings:

$$\iota_\infty : \bar{\mathbf{Q}} \hookrightarrow \mathbf{C}, \quad \iota_p : \bar{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$$

A CM-type Σ is a subset of K such that $I_K = \Sigma \bigsqcup \Sigma^c$.

1. The main conjecture for CM fields

Let K be a CM field; we fix a CM-type Σ for K and suppose that it induces a p-adic CM-type Σ_p, that is: if S_p is the set of p-adic places of K, we have $S_p = \Sigma_p \bigsqcup \Sigma_p^c$ where Σ_p is the set of p-adic places obtained by composing the elements of Σ with ι_p. Such a p-adic CM-type exists because of the hypothesis (ord).

Let $\mathfrak{c}$ be a non-trivial ideal of K prime to p; this will be the prime-to-p part of the conductor for our Hecke characters.

Let $K_{\mathfrak{c}p^m}$ be the ray-class field of K of conductor $\mathfrak{c}p^m$ and $K_{\mathfrak{c}p^\infty} = \bigcup_{m\geq 1} K_{\mathfrak{c}p^m}$. It is well-known that the Galois group $G_{\mathfrak{c}} = Gal(K_{\mathfrak{c}p^\infty}/K)$ is a $\mathbf{Z}_p$-module of rank $1+d+\delta$, with $\delta \geq 0$ (Leopoldt conjecture claims that $\delta = 0$, this is known for abelian number fields). We fix an isomorphism (not canonical in general) of $\mathbf{Z}_p$-modules:

$$G_{\mathfrak{c}} \simeq \Delta_{\mathfrak{c}} \times W \tag{1.0.1}$$

where W is free of rank $1+d+\delta$ over $\mathbf{Z}_p$ and $\Delta_{\mathfrak{c}}$ is finite. We can identify W with $Gal(K_\infty/K)$, where K_∞ is the composite of all the $\mathbf{Z}_p$-extensions of K. Of course, K_∞ is contained in $K_{\mathfrak{c}p^\infty}$, because $\mathbf{Z}_p$-extensions are unramified outside p.

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One can always write $K_{\mathfrak{c}p^\infty}$ as $K'K_\infty$, with K' a finite abelian extension of K such that $K' \cap K_\infty = K$, so that $\Delta_{\mathfrak{c}} \simeq Gal(K'/K)$.

The automorphism $c \in Gal(K/F)$ acts on $G_{\mathfrak{c}}$ as $g \mapsto g^c := \tilde{c}g\tilde{c}$, for any extension $\tilde{c}$ of c to $K_{\mathfrak{c}p^\infty}$. Since p is odd, we have an isomorphism of $\mathbf{Z}_p$-modules:

$$W \simeq W^+ \times W^-$$

with $W^\pm = (1 \pm c)W$. We fix a choice of topological generators $\gamma_1, ..., \gamma_{1+\delta}$ of W^+ and $\gamma_{2+\delta}, ..., \gamma_{1+d+\delta}$ of W^-; then we can identify $\mathbf{Z}_p[[G_{\mathfrak{c}}]]$ with the $\mathbf{Z}_p$-algebra $\mathbf{Z}_p[\Delta_{\mathfrak{c}}][[T_1, ..., T_{1+\delta}, S_1, ..., S_d]]$, via the maps $\gamma_i \mapsto 1 + T_i$, $i \leq 1 + \delta$ and $\gamma_i \mapsto 1 + S_i$ if $i > 1 + \delta$. We will assume that $T := T_1$ is the cyclotomic variable, corresponding to the generator γ_1 of the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of K. We also fix a p-adic unit $u \in \mathbf{Z}_p^*$ such that $\gamma_1\zeta = \zeta^u$, for any $\zeta \in \mu_{p^\infty}$.

Note that the exact sequence:

$$1 \to \Delta_{\mathfrak{c}}^+ \to \Delta_{\mathfrak{c}} \to \Delta_{\mathfrak{c}}^- \to 1$$

is not necessarily split.

We write $M_\Sigma(K_{\mathfrak{c}p^\infty})$ for the maximal abelian pro-p-extension of $K_{\mathfrak{c}p^\infty}$ unramified outside the places of $K_{\mathfrak{c}p^\infty}$ above Σ_p and we define:

$$X_{\Sigma,\mathfrak{c}} = Gal(M_\Sigma(K_{\mathfrak{c}p^\infty})/K_{\mathfrak{c}p^\infty}) \tag{1.0.2}$$

It is a module over the completed group algebra $\mathbf{Z}_p[[G_{\mathfrak{c}}]]$.

It is known that $X_{\Sigma,\mathfrak{c}}$ is a finitely generated and torsion $\mathbf{Z}_p[[W]]$-module, so that we can associate to $X_{\Sigma,\mathfrak{c}}$ a characteristic ideal $\mathcal{X}_{\Sigma,\mathfrak{c}}$ in $\mathbf{Z}_p[[W]]$. This means that there is a homomorphism of $\mathbf{Z}_p[[W]]$-modules $X_{\Sigma,\mathfrak{c}} \to \mathbf{Z}_p[[W]]/\mathcal{X}_{\Sigma,\mathfrak{c}}$ whose kernel and cokernel vanish when localized at primes of height ≤ 1. The characteristic ideal turns out to be a product of prime ideals of height 1, therefore it is principal. Remark that if we had chosen $M_\Sigma(K_{\mathfrak{c}p^\infty})$ unramified outside all the places over p, then $X_{\Sigma,\mathfrak{c}}$ would not be a torsion $\mathbf{Z}_p[[W]]$-module.

For a p-adic character ω of $\Delta_{\mathfrak{c}}$, we will write $\mathbf{Z}_p[\omega]$ for the finite abelian extension of $\mathbf{Z}_p$ generated by the values of ω, $X_{\Sigma,\mathfrak{c}}^\omega$ for the maximal quotient of $X_{\Sigma,\mathfrak{c}} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\omega]$ on which $\Delta_{\mathfrak{c}}$ acts trough ω, and $\mathcal{X}_{\Sigma,\mathfrak{c}}^\omega$ for its characteristic ideal in $\mathbf{Z}_p[\omega][[W]]$.

Let us fix a complete discrete valuation ring A contained in $\mathbf{C}_p$ and containing $\mathbf{Z}_p[\omega]$ for any character ω of $\Delta_{\mathfrak{c}}$. Put:

$$\Lambda_0 = A[[W]]$$

To the data $K, p, \Sigma, \mathfrak{c}, \omega$ we can associate a p-adic L function, following Katz, which we view as a power series $\mathcal{L}_{\Sigma,\mathfrak{c}}^\omega \in \Lambda_0$. This power series, as well as $X_{\Sigma,\mathfrak{c}}$, does depend on the choice of Σ; it is not known how it changes when we make a different choice.

We can now state the main conjecture:

Conjecture 1.0.1. *Suppose that ω is primitive outside p. Then the characteristic ideal $\mathcal{X}_{\Sigma,\mathfrak{c}}^\omega$ is generated by $\mathcal{L}_{\Sigma,\mathfrak{c}}^\omega$.*

This conjecture has been proved, by completely different methods, by K.Rubin in the quadratic imaginary case. More recently Hida, improving previous results with Tilouine, has proven the conjecture for the anticyclotomic variables and a quite general CM field. Our method permits to extend their result beyond the anticyclotomic part, though some restrictions are necessary (see theorem 3.0.1).

We use, in the proof of the theorem 3.0.1, the following theorem of A.Wiles, which establishes the main conjecture for totally real fields. Let F be a totally real field as before, ψ a p-adic Artin character of F and F_ψ the finite abelian extension of F attached to ψ, so that ψ is a faithful representation of $Gal(F_\psi/F)$. We assume that ψ is odd, that is: $\psi(c)=-1$. Let M_∞ be the maximal unramified abelian pro-p-extension of $F_\psi F_\infty$ (F_∞ is the cyclotomic $\mathbf{Z}_p$-extension of F); then $Y = Gal(M_\infty/F_\psi F_\infty)$ is a finitely generated $\mathbf{Z}_p[[Gal(F_\psi F_\infty/F)]]$-module. If we assume $F_\psi \cap F_\infty = F$, we have an isomorphism of $\mathbf{Z}_p$-modules: $Gal(F_\psi F_\infty/F) \simeq \Delta_F \times W_0$, with $\Delta_F = Gal(F_\psi/F)$ and $W_0 \simeq \mathbf{Z}_p$. It is known that the eigenspace Y^ψ, corresponding to the action of Δ_F via ψ, admits a characteristic ideal $\mathcal{Y}^\psi$. Deligne and Ribet proved the existence of a power series $G^\psi(T) \in \mathbf{Z}_p[\psi][[T]]$ such that

$$G^\psi(u^n-1) = L(1-n,\psi)\prod_{\mathfrak{p}|p}(1-\psi\tau^{-n}(\mathfrak{p})N\mathfrak{p}^{n-1}) \tag{1.0.3}$$

for any integer $n \geq 1$, where u is fixed as before and $\tau : Gal(F(\zeta_p)/F) \to \mu_{p-1}$ is the Teichmüller character.

For a general ψ, one can write $\psi = \chi\rho$, with $F_\chi \cap F_\infty = F$ and $F_\rho \subset F_\infty$. One has then: $G_\psi(T) = G_\chi(\rho(\gamma)(1+T)-1)$, for a fixed topological generator γ of W_0.

Put $\psi^\vee = \psi^{-1}\tau$. Then, Wiles proved:

Theorem 1.0.2. *Let ψ be odd and define: $\mathcal{L}^\psi(T) := G^{\psi^\vee}(u(1+T)^{-1}-1)$. Then, if p is odd, $\mathcal{L}^\psi(T)$ generates $\mathcal{Y}^\psi$.*

Note that KF_∞ is the cyclotomic $\mathbf{Z}_p$-extension of K and the restriction map from $Gal(KF_\infty/K)$ to $Gal(F_\infty/F)$ is an isomorphism, therefore a topological generator of $Gal(F_\infty/F)$ gives in a natural way a topological generator of $Gal(KF_\infty/K)$. In §5, we will fix an embedding of $Gal(F_\infty/F)$ into W.

2. The Eisenstein ideal

2.1. **Unitary groups.**

Let $A \in GL_3(K)$ be a matrix such that $A^c = -{}^tA$ and put:

$$GU(A) = \{g \in GL_3(K); gA^tg^c = \nu(g)A\}$$

We call $GU(A)$ the unitary group associated to A, it is a reductive group defined over F.

We write $GU(2,1) = GU(J)$, where:

$$J = \begin{pmatrix} & & 1 \\ & \vartheta & \\ -1 & & \end{pmatrix} \tag{2.1.1}$$

and ϑ is an element of K with trace zero over F, such that $Im(\vartheta^\sigma) > 0$ for all $\sigma \in \Sigma$. Then $GU(2,1)$ is quasi-split at all infinite places of F. The kernel of ν is denoted by $U(2,1)$ and it is a semi-simple group defined over F.

We write P for the standard parabolic subgroup of upper triangular matrices in $G = GU(2,1)$, with Levi subgroup M and unipotent radical U. We use the notation $d(z,x) = zdiag(x^c, 1, x^{-1})$ for the elements of M. The center Z of G is isomorphic to $R_{K/F}\mathbf{G}_m$.

2.2. **Eisenstein series: outline of the strategy.** To a Hecke character χ of K^*, one can associate a holomorphic Eisenstein series E_χ on $U(2,1)$, of weight k, whose constant term is given by the product of L-functions:

$$L(k,\chi)L(2k,\chi_F\tau) \tag{2.2.1}$$

We refer to Shimura book, 'Euler products and Eisenstein series', for general backgroud on these Eisenstein series, and on automorphic forms on unitary groups.

Our plan for the proof of the main conjecture is the following:

(1) normalize and interpolate p-adically the series E_χ. The resulting p-adic modular form will have constant term equal to $\mathcal{L}_\Sigma^\omega\mathcal{L}^\psi$. We allow χ to vary in the set of algebraic Hecke characters of K^*, unramified outside $\mathfrak{c}p$, critical at 0 (a condition on the infinity type) and such that the restriction to $\Delta_\mathfrak{c}$ of the associated Galois character is ω.

(2) prove that such a p-adic Eisenstein series doesn't vanish modulo its constant term. Here the major difficulty is the intrinsec complexity of the Fourier-Jacobi expansion.

(3) introduce an appropriate Eisenstein ideal, say Eis_Σ^ω, supposed to measure the congruences between (p-adic) cusp forms and the p-adic Eisenstein series on $U(2,1)$. The theory of p-adic automorphic forms on unitary groups (actually, on reductive groups) has been worked out, independently, by D.Mauger and H.Hida in recent years. The Eisenstein ideal is an ideal of the universal nearly-ordinary Hecke algebra of $GU(2,1)$ which is finite and flat over an Iwasawa algebra in $1+3d+\delta$ variables, containing Λ_0 in a natural way.

(4) prove the divisibilities:

$$\mathcal{L}_\Sigma^\omega\mathcal{L}^\psi | Eis_\Sigma^\omega | \mathcal{X}_\Sigma^\omega\mathcal{L}^\psi \tag{2.2.2}$$

(on the right we used Wiles' theorem). We will see that the can only prove a weaker divisibility, the problem being related to the existence of cuspidal endoscopic forms coming from $U(1,1)$.

In the following, we will try to explain in more detail some of these steps, but we won't discuss the divisibility $\mathcal{L}_\Sigma^\omega\mathcal{L}^\psi | Eis_\Sigma^\omega$ in this report.

2.3. **p-adic Eisenstein series.** The basic idea is to use a pullback formula, orignally due to Shimura. We fix an embedding of $U(2,1)\times U(1)$ in $U(2,2)$. If $\tilde{E}$ is an Eisenstein series on $U(2,2)$, of Siegel type, induced by the character χ, we can write:

$$\tilde{E}(g) = \tilde{E}(g;s,\phi) = \sum_{\gamma\in P'\backslash U(2,2)} \phi(\gamma g) \tag{2.3.1}$$

if P' is the Siegel parabolic and ϕ is a section of the (normalized) induced representation $Ind_{P'}^{U(2,2)}\chi|\,|^s$. This series converges, as usual, for $Re(s)$ sufficiently large and has a meromorphic continuation to $\mathbf{C}$. Our choice of ϕ_∞ is such that $\tilde{E}$ is holomorphic at $s=1/2$.

Fix a character λ of $U(1)(\mathbf{A})$. An easy computation gives:

$$\int_{G_2\backslash G_2(\mathbf{A})} \tilde{E}(g_1,g_2)\lambda(g_2)dg_2 = \int_{G_2\backslash G_2(\mathbf{A})} \sum_{\gamma\in P'\backslash U(2,2)} \phi(\gamma(g_1,g_2))\lambda(g_2)dg_2 =$$

$$\int_{G_2\backslash G_2(\mathbf{A})} \sum_{\gamma_1\in P_1\backslash G_1} \sum_{\gamma_2\in G_2} \phi(\gamma_1 g_1,\gamma_2 g_2))\lambda(g_2)dg_2 = \sum_{\gamma_1\in P\backslash G_1} \int_{G_2(\mathbf{A})} \phi(\gamma_1 g_1, g_2)\lambda(g_2)dg_2$$

where $G_2 = U(1), G_1 = U(2,1)$. The last series is easily recognized as an Eisenstein series on $U(2,1)$. We have used the decomposition:

$$U(2,2) = P'(U(2,1)\times U(1)) \tag{2.3.2}$$

It is important to observe that the last integral is actually a finite sum, so that the p-integrality for the last Eisenstein series follows easily from the p-integrality for $\tilde{E}$.

Now, the point is to choose ϕ such that $\tilde{E}(g;s,\phi)$ is p-integral at $s=1/2$ and to compute the resulting Eisenstein series on $U(2,1)$. The first problem has been essentially settled by Harris, Li and Skinner in a recent joint work. The consequent computations for $U(2,1)$ will be, hopefully, made public soon by myself.

The section ϕ is to be chosen as follows: let S be the union of the infinite places, the places above p and the places where χ or K/F ramify. Then one defines $\phi=\otimes\phi_v$ such that ϕ_v is the normalized spherical vector in $I(s,\chi)$ for $v\notin S$, ϕ_v is holomorphic at $s=1/2$ if v is infinite, ϕ_v for $v|p$ has support in the big cell of the Bruhat decomposition and is designed in order to give the right Euler factors in the constant term. As we said, the choice of ϕ_v at ramified places is insensitive to the p-adic variation of χ (this is unsatisfactory and it will be dealt with in near future). With these choices, the Eisenstein series $\tilde{E}$, as well as the Eisenstein series on $U(2,1)$ have zero constant term at infinity. We recover the constant term at another (highly ramified) cusp.

The Fourier coefficients of $\tilde{E}$ at the unramified places were calculated by Shimura in his book (*loc.cit.*).

We note that this method works certainly for any unitary group of the form $U(n,1)$, using the embedding $U(n,1)\times U(n-1)\hookrightarrow U(n,n)$.

As a final remark we note that this method requires a compatibility between χ and λ, plus the assumption that the weight of $\tilde{E}$ is o scalar type. In order to get rid of these restrictions, one has to apply a p-adic version of Maass' differential operators, as was done by Katz in his paper on p-adic L-functions for CM fields.

Regarding the non-vanishing modulo the constant term, we content ourselves to say that one can combine the computations of primitive Fourier-Jacobi coefficients (cf. Murase, Sugano, J.Math.Sci.Univ.Tokyo 9), with Hida's theorem on the non-vanishing modulo p of the special values of Hecke L-series in towers of anticyclotomic characters with prime-to-p conductor.

3. From congruences to cocycles

The proof of the next theorem is independent of what we explained in the previous section; in fact, in order to define Eis^{ω}_{Σ} it suffices to interpolate p-adically the

Hecke eigenvalues of the Eisenstein series E_χ, which is clearly easier than interpolate p-adically the Eisenstein series itself.

In order to state our theorem we need to introduce some notation. Let D_w the decomposition subgroup of $G_{\mathfrak{c}}$ at a place w over p; it is a closed subgroup of finite index in $G_{\mathfrak{c}}$. The splitting 1.0.1 induces an isomorphism: $D_w \simeq D_w^{tors} \times D_w^{free}$, with $D_w^{tors} \subset \Delta_{\mathfrak{c}}$ and $D_w^{free} \subset W$.

We associate a p-adic character of finite order ψ of F, to any character ω of $\Delta_{\mathfrak{c}}$; it corresponds to the fixed torsion of the Galois character associated to $\chi_F\tau$, with χ varying as explained before.

Our main result is:

Theorem 3.0.1. *Let p be odd and let P be a prime ideal of Λ_0 of height 1. Let $\mathfrak{c}$ be an ideal of K such that $\mathfrak{c}^c = \mathfrak{c}$ and assume $\Delta_{K/F} \nmid \mathfrak{c} \cap O_F$. Suppose:*

(1) *$p \nmid \varphi(N\mathfrak{c})$ or $P \neq (p)$;*
(2) *$P \mod J \neq (0)$ and $P \mod J \nmid \mathcal{L}^\psi(T)$;*
(3) *ω is a p-adic character of $\Delta_{\mathfrak{c}}$, which is primitive outside p.*
(4) *if $\delta = 0$ and $\omega\omega^c$ is trivial when restricted to D_w^{tors} at a place w of Σ_p, then $u(1+T) - \zeta \notin P$, for every $\zeta \in \mu_p$.*

Then, we have:

$$v_P(\mathcal{X}^\omega_{\Sigma,\mathfrak{c}}\mathbf{B}) \geq v_P(Eis^\omega_\Sigma) \tag{3.0.3}$$

Here $\mathbf{B}$ is an Iwasawa algebra in $1 + 3d + \delta$ variables containing Λ_0 and Eis^ω_Σ is an ideal of $\mathbf{B}$, see §3.2.

Remark 3.0.2. Actually, we could state the theorem in a slightly more general form: instead of the assumption (2), we could assume that P does not divide the characteristic ideal of another Iwasawa module (associated to $\omega\omega^c$) which, when $P \mod J \neq (0)$, can be compared to the p-adic L-function $\mathcal{L}^\psi$ by means of theorem 1.0.2.

We want to stress that the more serious hypothesis in this theorem is the second one (or its weaker form, as just explained), and it will be probably necessary to introduce new ideas in order to deal with the cases where P divides $\mathcal{L}^\psi$.

The hypothesis 1,3 and 4 are analog to the assumptions in lemma 6.1 of Wiles'proof and it should be possible, in principle, to remove them. Note that the condition $p \nmid \varphi(N\mathfrak{c})$ appears also in Hida & Tilouine's theorem.

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