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pg 1

Minimal modularity lifting theorems for imaginary quadratic fields

Joint work with D. Geraghty

Motivating Problem

$F = \#$ field

$\rho : G_F \to G(\overline{\mathbb{Q}}_p)$ $\quad G = GL_n$ here

Assume that

- ρ is unram. outside finitely primes
- $\rho|_{D_{F,v}}$ for $v|p$ is potentially semi-stable.

One wants to deduce that ρ is modular (in some sense).

One could consider the situation

$$\begin{tikzcd} \rho : G_F \arrow[r] \arrow[dr, dashed] & GL_n(\overline{\mathbb{Q}}_p) \\ & G(\overline{\mathbb{Q}}_p) \arrow[u, phantom, "\cup"] \end{tikzcd}.$$

Choose a lattice $\Rightarrow$ get residual rep. $\bar{\rho}$.

"Numerical criterion":

$$\underbrace{\text{(Expected dim. of universal deformation space of } \bar{\rho})}_{\text{given by Galois cohom.}}$$

$-$ (codimension in local $\overset{(v|p)}{\text{deformation}}$ ring cut out by imposing that lift is potentially semistable of some fixed type) $= 0$. $\Big\}$ compute in terms of G

LHS = relative dim of $R_{\bar{\rho}} / \mathbb{Z}_p$

RHS = relative dim of $\mathbb{T}$ = Hecke ring.

$\bar{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\bar{\mathbb{F}}_p)$

- if $\det \bar{\rho}(c) = -1$, $3-3=0$
- if $\det \bar{\rho}(c) = 1$, $1-3 \neq 0$.

Suppose now we look at $\rho : G_F \to GL_2(\bar{\mathbb{Q}}_p)$. Let F have signature (r_1, r_2). Assume $\bar{\rho}(c_v)$ has det $= -1$ if $v | \infty$, $F_v = \mathbb{R}$.

$$\underbrace{}_{\text{ex}} - \underbrace{}_{\text{loc}} = -r_2 \qquad \text{(from above)}$$

G a group, authentic quotients of the symmetric space.

Γ lattice in G

$H^*_{\mathcal{B}}(\Gamma \backslash G^X, \mathbb{R})$ interesting coh. classes $\leftrightarrow \pi$, π_∞ tempered

The tempered π contribute to cohomology in degrees $q_0, \ldots, q_0 + l_0$ $\quad l_0 = \mathrm{rk}\, G - \mathrm{rk}\, K - \mathrm{rk}\, A^\circ_{\mathbb{Q}}$.

Expect $\underbrace{}_{\text{ex}} - \underbrace{}_{\text{loc}} = -l_0$.

Want to focus on the case that $l_0 = 1$.

$\Rightarrow \dim X = 2q_0 + 1 \equiv 1 \pmod 2$

X real manifold with no complex structure.

We can now restrict to a special case:

$\rho : G_F \longrightarrow GL_2(\bar{\mathbb{Q}}_p)$

$F =$ imag. quad. field, $\quad X = \mathbb{H}^3$

$\Gamma \subseteq GL_2(\mathcal{O}_F)$. Assume for simplicity that $h_F = 1$.

$H_i(X/\Gamma, \mathbb{Z})$ are not torsion free in general.

$H_i(X/\Gamma, \mathbb{F}_p) \circlearrowleft \mathbb{T}$ via double cosets. as usual

<u>Conjecture:</u> Given a maximal ideal m of $\mathbb{T}$, $\exists$

$$\rho : G_F \longrightarrow GL_2(\mathbb{T}/m)$$

s.t.

$$\mathrm{tr}(\rho(\mathrm{Frob}_\lambda)) = T_\lambda \pmod{m}.$$

<u>Conjecture (Serre's Conj):</u> Given $\bar{\rho} : G_F \to GL_2(\overline{\mathbb{F}}_p)$ irred. $\Rightarrow \exists\ \Gamma, V,\ m \subseteq \mathbb{T}$ acting on $H_i(\Gamma\backslash X,$ [illegible]$)$. ?
is the Galois rep.

3-manifold:

H_2	H_1
$\mathbb{Z}^r$	$\mathbb{Z}^r + T$

$\uparrow$ interesting Galois reps.

$\mathbb{T} \circlearrowright H_1(\Gamma\backslash X, \mathbb{Z})$

Let m be a maximal ideal of $\mathbb{T}$.

Conj. A: $\exists\ \rho : G_F \longrightarrow GL_2(\mathbb{T}_m)$ satisfying local-global compatibility, $\mathrm{tr}(\rho(\mathrm{Frob}_x)) = T_x$.

Let $\bar\rho = \bar\rho_m \rightsquigarrow R_{\bar\rho}$ minimal deformation ring.
Assume level Γ is coprime to p.

Conj. A $\Rightarrow \exists$ a surj. map $R_{\bar\rho} \to \mathbb{T}_m$.

Thm (C.-Geraghty): Assume conj. A. Assume $\bar\rho|_{G_{F(\zeta_p)}}$ is irred, $p \geq 3$.
(• multiplicity 1 hypothesis $\Rightarrow R_{\bar\rho} \xrightarrow{\sim} \mathbb{T}_m$)
~~Given~~ Let ~~$\mathbb{T}_m \to K$, char K = 0~~,

$$\begin{tikzcd} \mathbb{T}_m \arrow[r] & K \\ R_{\bar\rho} \arrow[u] \arrow[ur] & \end{tikzcd}$$

char 0 · field K

$$\begin{tikzcd} & & \mathbb{Z}_p[[S_1,\dots,S_n]] \\ \mathbb{Z}_p[[x_1,\dots,x_n]] \arrow[r] & R_\infty \arrow[r] \arrow[d] & \mathbb{T}_\infty \arrow[d] \\ & R_{\bar\rho} \arrow[r] & \mathbb{T}_m \end{tikzcd}$$

usual Taylor-Wiles method.

In this case

$$\begin{tikzcd} \mathbb{Z}_p[[x_1,\dots,x_{n-1}]] \arrow[r] & R_\infty \arrow[r] \arrow[d] & \mathbb{T}_\infty \arrow[d] \\ & R_{\bar\rho} \arrow[r] & \mathbb{T}_m \end{tikzcd}$$

Choose primes $q \equiv 1 \pmod{p^n}$ s.t. $\bar\rho(\mathrm{Frob}_q)$ has distinct eigenvalues.

$\Rightarrow$

$$H_1(X/\Gamma_1, \mathbb{Z}_p)_{\mathfrak{m}} = H_1(X/\Gamma_0(q), \mathbb{Z}_p)_{\mathfrak{m}}$$

$$\underbrace{\Gamma_0(q) \supseteq \Gamma_H(q)}_{\Delta \text{ cyclic of order } p^n} \supseteq \Gamma_1(q)$$

$$H_1(X/\Gamma_H(q), \mathbb{Z}_p)_{\mathfrak{m}}$$

$$H_1(X/\Gamma_0(q), \mathbb{Z}_p[\Delta]).$$

There is an exact seq.

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_p[\Delta] \to \mathbb{Z}_p[x]/?$$