The Proof of the Burger-Sarnak Conjecture

Spectrum and cohomology of locally hyperbolic varieties:
(Some applications of Arthur's results)
(Joint work w/ Nicolas Bergeron)

1. The Burger-Li-Sarnak Conjecture:

G = Group over $\mathbb{Q}$, quasi simple
$G(\mathbb{R}) = SO(n,1) \times SO(n+1)^{d-1}$
(start with ~~G_F~~ G_F = group of orthogonal type over F totally real, then $G = \mathrm{Res}_{F/\mathbb{Q}}(G_F)$..)

Note: The type of G_F as a group over a number field may be dual (not nec. a true orthogonal group)

But: Assume G is not a ~~triality~~ triality group

Now consider
- $\Gamma \subset G(\mathbb{Q})$ congruence subgroup
- $\mathcal{H}^n = SO(n,1)/SO(n)$
- $S_\Gamma = \Gamma \backslash \mathcal{H}^n$

Problem: Spectrum of S_Γ?

Natural Laplacian ω (normalization: $\left(\frac{n-1}{2}\right)^2$ is the bottom spectrum for $L^2(\mathcal{H}^n)$)

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<u>Note:</u> $SO(2,1)$ "is" $SL(2,\mathbb{R})$ and so we get $\frac{1}{4}$, which agrees w/ Selberg's conj.

- ω is the positive Laplacian (locally it is $-\sum \frac{\partial^2}{\partial x_i^2} + \cdots$)

Burger-Li-Sarnak '92, Sarnak Kyoto '90: Conjecture ~~of~~ on ~~the~~ spectrum of ω:

<u>Conjecture:</u> The spectrum of ω is contained in the union

$$\left[\left(\tfrac{n-1}{2}\right)^2, \infty\right[\quad \text{(tempered spectrum)}$$

$$\left\{\left(\tfrac{n-1}{2}\right)^2 - \left(\tfrac{n-1}{2} - j\right)^2\right\} \quad j = 0, 1, \ldots, \left\lfloor \tfrac{n-1}{2} \right\rfloor$$

where $\lfloor x \rfloor$ = largest integer $< x$.

<u>Note:</u> if we go back to $\Delta = -\omega$

bottom spectrum

excited states

Bergeron (thesis ~ 2000) for topological applications, one wants to extend the conjecture to Hodge Laplacian ω_k on k-forms. (Pertinent point: get a nontrivial spectral gap) (2005 in book)

Write $\delta_N = \frac{1}{2} - \varepsilon_N$, $\varepsilon_N = \frac{1}{N^2+1}$ $\quad (\delta_N < \frac{1}{2})$

(Recall: ε_N is the Luo-Rudnick-Sarnak improvement on the Jacquet-Shalika "distance to purity $< 1/2$" estimate. Ramanujan conj. $|t_i| = 1$. Jacquet-Shalika $q^{-1/2} < |t_i| < q^{1/2}$. L-R-S $\frac{1}{2} \to 1/2 - \varepsilon_N$ here.)

<u>Assume:</u> $0 \le k \le \frac{n}{2}$ (n even)

$0 \le k < \frac{n-1}{2}$ (n odd)

<u>Theorem:</u> For a congruence gp. as above, the spectrum of ω_k on co-closed forms is contained in the union of

$$\left[\left(\tfrac{n-1}{2} - k\right)^2 - \delta_N^2, \infty\right[$$

$$\left\{\left(\tfrac{n-1}{2} - k\right)^2 - \left(\tfrac{n-1}{2} - j\right)^2\right\}, \quad k \le j \le \tfrac{n-1}{2}$$

Here N is described by Langland's functoriality:

$$G = SO(n,1) \qquad \hat{G} = \begin{cases} SO(n+1, \mathbb{C}) & n+1 \text{ even} \\ Sp(2l, \mathbb{C}) & n = 2l \end{cases}$$

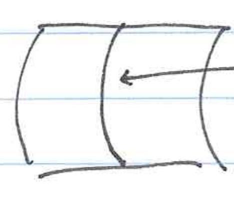
so $N = \begin{cases} n+1 & n+1 \text{ even} \\ n & n \text{ even} \end{cases}$.

2. Topological Consequences:

Quasi-Lefschetz properties {Harris-Li, Venkatesh (unitary groups) also $\mathcal{H}^n$ (no complex) }

X = Symm sp

$\Gamma \backslash X$



$H = \Delta \backslash Y$

$$H^\bullet(\Gamma\backslash X) \xrightarrow{?} H^\bullet(\Delta\backslash Y).$$

G "orthogonal"

$H \subset G \quad /\mathbb{Q}$

$$H(\mathbb{R})/K_H \hookrightarrow G(\mathbb{R})/K_G \cong \mathcal{H}^n$$

$\quad \wr\wr$

$\quad \mathcal{H}^m$

$\Rightarrow \quad S_{H,\Delta} < S_{G,\Gamma}$

(everything cocompact is an assumption here

$$H^\bullet(S_{G,\Gamma}) \longrightarrow H^\bullet(S_{H,\Delta}).$$

Also $\forall\ \gamma \in G(\mathbb{Q})$

$$H^\bullet(S_{G,\Gamma}) \longrightarrow H^\bullet(S_{H, H\cap\gamma^{-1}\Gamma\gamma}).$$

Thm: The "virtual restriction"

$$H^k(S_{G,\Gamma}) \to \prod_{\gamma \in G(\mathbb{Q})} H^k(S_{H,\Gamma^\gamma})$$

is injective for $k \le \frac{m}{2}$, $m =$ dim.

Remarks: 1) Different proof (Bergeron, Haglund & Wise)

2) Same should be true for hyperbolic complex varieties.

3) The spectral conjectures are absolutely limited to congruence groups.

(Conjecture $\Gamma \subseteq SO(n,1)$ For some subgroup
$\Delta \subseteq \Gamma$, $h^1(\Delta, \mathbb{C}) \neq 0$ (1^{st} Betti #).
i.e, $\Delta \twoheadrightarrow \mathbb{Z}$)

3. Exotic varieties in dimension 7:

Note: Conjecture known for arithmetic groups, $n \neq 3, 7$.

Dim 7 appears because of triality.

$\mathbb{Q} = F$

Recall: $G/\mathbb{Q}$ defined by

1) Dynkin diagram $G/\overline{\mathbb{Q}}$ =  Dyn.
& action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on Dyn.

2) Take an inner form

A triality group is one for which Gal acts by the full S_3. (The quotients are compact)

$\Gamma \backslash \mathcal{H}^7$.

Thm: Assume Γ comes from a triality group G_F. Then for any congruence subgroup $\Gamma \subset G(\mathbb{Q})$
$$H^1(\Gamma \backslash \mathcal{H}^7) = H^1(\Gamma, \mathbb{C}) = \{0\}.$$

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<u>Remarks:</u> 1) Given S_Γ, there covers of arbitrarily large degrees so that $h^1(S_\Delta) = 0$ $(\Delta \subset \Gamma)$

2) We do not know that all arithmetic groups are congruence groups. Thurston's conj. might still be true.