Moduli Stacks of local Galois representations:

Joint project w/ Toby Gee

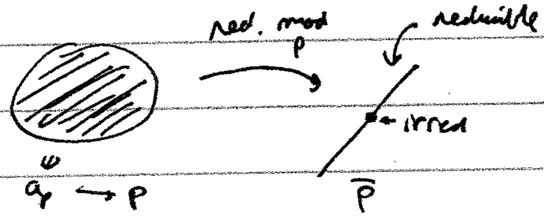
Idea: To make moduli spaces of p-adic & mod p reps. of G_K where $K/\mathbb{Q}_p$ is finite.

Mazur, et. al: Formal moduli of such reps. We want actual algebraic moduli. (so ρ really varies in families)

One motivation: reduction of crystalline Galois representations

If $K = \mathbb{Q}_p$, $\rho: G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{Q}}_p)$ cts and crys. w/ H.T weights $0, k-1$, $\det = \text{cycl}^{1-k}$

ρ is (essentially) classified by a_p = tr. of crystalline Frob.



formal $\mathbb{A}^1/\mathbb{Z}_p$

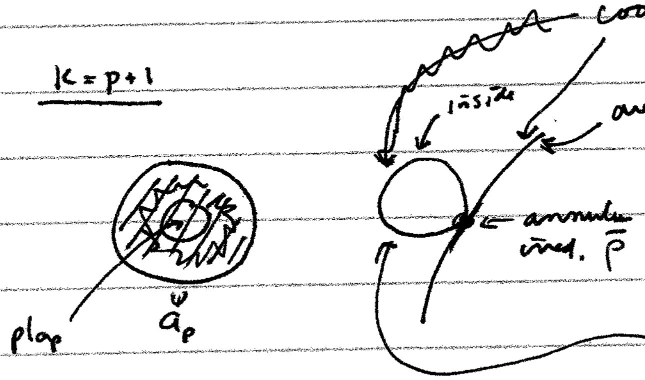
$2 \le k \le p$

$a_p \longmapsto \bar a_p.$

<u>$k = p+1$</u>

coordinate here is $\overline{(a_p/p)}$.

one ext $\begin{pmatrix} \text{unr.} & * \\ 0 & \text{unr.}\cdot\overline{\text{cycl}}^{1-k} \end{pmatrix} \longleftrightarrow \mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p^{\,2}$



formal $\mathbb{A}^1/\mathbb{Z}_p$ blown up at a pt in the special fibre.

ext. $\overline{\text{cycl}}^{-1} \otimes \begin{pmatrix} \text{unr} & * \\ 0 & \text{unr} \end{pmatrix} \leftrightarrow \mathrm{Sym}^{\overset{?}{*}}\,\overline{\mathbb{F}}_p^{\,2} \otimes \det^{-1}$.

<u>Goal:</u> For any $K/\mathbb{Q}_p$ finite, type τ, integer h, to construct a formal ~~adic~~ stack $/\mathbb{Z}_p$ (top. fin. type)

$$\mathcal{X}^{\tau,h}$$

$\overline{\mathbb{Z}}_p$-pts $\longleftrightarrow$ $\rho : G_K \to GL_n(\overline{\mathbb{Z}}_p)$ s.t.

$\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is pot. semi-stable with wt. $\in [0,h]$ and type τ.

And also construct $\overline{\mathcal{X}}$ a finite type alg. stack $/\mathbb{F}_p$ s.t. $\overline{\mathbb{F}}_p$-pts of $\overline{\mathcal{X}}$ $\longleftrightarrow$ $\bar{\rho} : G_K \to GL_n(\overline{\mathbb{F}}_p)$ and give an embedding $(\overline{\mathcal{X}}^{\tau,h})_{red} \hookrightarrow \overline{\mathcal{X}}$ whose image is a union of components s.t. the specialization map

$$\mathcal{X}^{\tau,h}(\overline{\mathbb{Z}}_p) \xrightarrow{sp} (\overline{\mathcal{X}}^{\tau,h}_{red})(\overline{\mathbb{F}}_p) \hookrightarrow \overline{\mathcal{X}}(\overline{\mathbb{F}}_p)$$

is $\rho \mapsto \bar{\rho}$.

If h is small and $K/\mathbb{Q}_p$ unr., one can make components of $\overline{\mathcal{X}}$ as moduli space of F-L modules. (Daniel Le's thesis)
If h isn't small, this doesn't work.

$K=\mathbb{Q}_p$, GL_2, $h=p-1$: fix det = trivial, then the reducible $\bar{\rho}$ are in ~~[illegible]~~ a family

$$\begin{pmatrix} unr_x & * \\ 0 & unr_{x^{-1}} \end{pmatrix}$$

So we have the following picture:

Emerton
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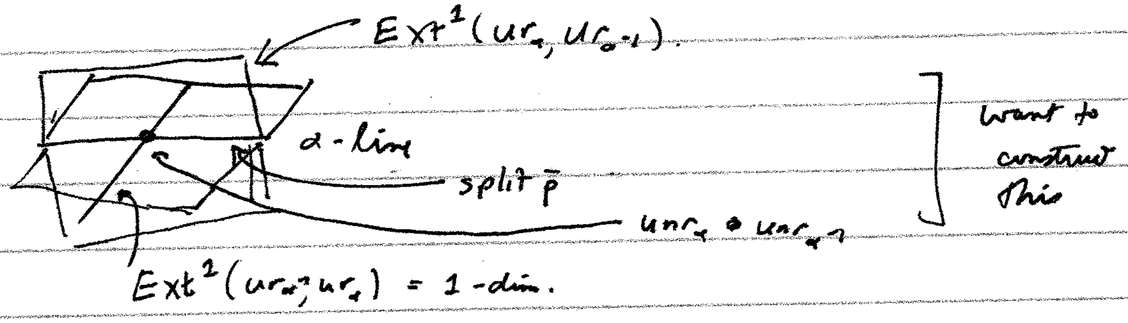
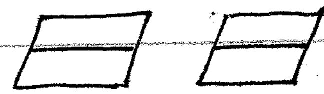
PS3



F-L. theory:
produces,



so the functor to Galois reps is not faithful.

So we need to glue them somehow along this line.

We have good progress towards the construction of $\mathcal{X}$, $\bar{\mathcal{X}}$, etc.
The details are worked out so far after replacing K by K_∞, where $K_\infty = \bigcup_{n\geq 1} K(\pi^{p^{-n}})$.
This is enough to handle the case of $h \leq 1$.
To get the general case, we will have to add $\hat{G}$-structure following Tong Liu.

Over K_∞, Fontaine-Breuil-Kisin give a nice integral p-adic Hodge theory, which Pappas-Rapoport put in moduli.
Work over $\mathbb{Z}/p^a\mathbb{Z}$, a fixed. Fix d.

$$\mathcal{C}^{\leq h} \longrightarrow \mathcal{R} \qquad \mathfrak{M} \longmapsto \mathfrak{M}[\tfrac{1}{u}]$$

$\mathcal{C}^{\leq h}$: moduli stack of B-K modules of height $\leq h$

$\mathcal{R}$: moduli stack of étale φ-modules

M - $A((u)) \otimes_{W(k)}$-module loc. free rk d

A given $\mathbb{Z}/p^a\mathbb{Z}$-alg.

$\mathfrak{M}/A[[u]] \otimes_{\mathbb{Z}_p} W(k)$ locally free of rk d. $\quad \varphi^*M \to M$

$\varphi^* \mathcal{M} \longrightarrow \mathcal{M}^p$

coherent killed by Elsh.

Build $\overline{\mathcal{X}}$ as the union of the image of these maps.

We need to construct these images as objects in alg. geo.

(right now they exist as points on a case by case basis)

What we have is

$$\mathcal{C}^{sh} \longrightarrow \mathcal{R} \quad \text{is proper}$$

$\mathcal{C}^{sh}$: alg. stack of fin. type$/\mathbb{Z}/p^a\mathbb{Z}$

$\mathcal{R}$: huge, $\Delta : \mathcal{R} \to \mathcal{R} \times \mathcal{R}$ is representable, of finite type.

$\mathcal{R}$ admits versal rings at each fin. pt.

Set-up: $\mathcal{X} \xrightarrow{\xi} \mathcal{F}$ stack$/S_{et}$

$\mathcal{X}$: alg. stack of f.t.$/S$

$\mathcal{F}$: $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is rep. by alg. spaces loc. f.p.

$\mathcal{F}$ has versal rings at each f.t. pt

Define: "scheme-theoretic image" of ξ. $Z \hookrightarrow \mathcal{F}$ substack

- If A is local Artinian f.t.$/S$, $Z(A) \subseteq \mathcal{F}(A)$ are the maps $\operatorname{Spec} A \hookrightarrow \mathcal{F}$ which factor as $\operatorname{Spec} A \to \operatorname{Spec} B \to \mathcal{F}$ where
  - B is complete local Noeth$/S$.
  - $\mathcal{X}_B \to \operatorname{Spec} B$ is sch. th. dominant

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathcal{F} \end{array} \qquad \begin{array}{c} \mathcal{X}_B \\ \downarrow \\ \operatorname{Spec} B \end{array}$$

- If T/S is finite type, $Z(T) \hookrightarrow \mathcal{F}(T)$ to be maps s.t. $\operatorname{Spec} \mathcal{O}_{T,t}/\mathfrak{m}_t^n \longrightarrow \mathcal{F}$ lie in Z $\forall\ n \geq 1$, all t a fin. type pt in T

- If T/S is affine, write $T = \varprojlim T_i$ w/ T_i of fin type$/S$ and define ~~$Z(T) = \varinjlim Z(T_i)$~~

$$Z(T) = \bigcup_i \left(\operatorname{im} Z(T_i) \to \mathcal{F}(T_i) \to \mathcal{F}(T) \right).$$

<u>Thm:</u> If $\xi : \mathcal{X} \hookrightarrow \mathcal{F}$ is proper, and $\mathcal{X}$ is an alg. stack of fin. type$/S$, $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is rep. representable by alg. space lfp. and $\mathcal{F}$ admit versal ring at each fin. pt, Z admits effective versal rings at fin pts, then Z is an alg. stack loc. fin type$/S$.