Cycles on Shimura varieties and applications to Faltings heights

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Colmez's Conjecture

$$\begin{tikzcd} E \arrow[d, dash, "2"] & \text{CM field} \\ E^+ \arrow[d, dash, "n"] & \\ \mathbb{Q} & \end{tikzcd}$$

$\Phi \subset \mathrm{Hom}(E,\mathbb{C})$ s.t. $\mathrm{Hom}(E,\mathbb{C}) = \Phi \sqcup \overline{\Phi}$.

$\Phi = \{\varphi_1,\dots,\varphi_n\}$

Suppose $A/\mathbb{C}$ is an abelian variety with $\mathcal{O}_E \to \mathrm{End}(A)$. and type Φ, i.e. $x \in \mathcal{O}_E$ acts on $\mathrm{Lie}(A) \cong \mathbb{C}^n$ as

$$\begin{pmatrix} \varphi_1(x) & & \\ & \ddots & \\ & & \varphi_n(x) \end{pmatrix}.$$

Fix a number field L large enough so that A is defined with good reduction over L.

$A \xrightarrow{\pi} \mathrm{Spec}(\mathcal{O}_L)$.

Line bundle $\det(\pi_* \Omega^1_{A/\mathcal{O}_L}) \in \mathrm{Pic}(\mathrm{Spec}(\mathcal{O}_L))$.

Pick a rational section ω

$$h_{\infty}^{Falt}(A,\omega) = \frac{-1}{2[L:\mathbb{Q}]} \sum_{\sigma: L \hookrightarrow \mathbb{C}} \log \left| \int_{A^{\sigma}(\mathbb{C})} \omega^{\sigma} \wedge \overline{\omega}^{\sigma} \right|$$

$$h_{f}^{Falt}(A,\omega) = \frac{1}{[L:\mathbb{Q}]} \sum_{\mathfrak{p} \subseteq \mathcal{O}_L} \mathrm{ord}_{\mathfrak{p}}(\omega) \log N(\mathfrak{p}) .$$

The Faltings height $h^{Falt}(A) = h_{\infty}^{Falt}(A,\omega) + h_{f}^{Falt}(A,\omega)$ depends only on $A/\mathbb{C}$.

Thm (Colmez): $h^{Falt}(A)$ depends only on (E,Φ). Call it $h^{Falt}(E,\Phi)$.

$\underset{\substack{\rotatebox[origin=c]{90}{$\in$} \\ \sigma}}{G_{\mathbb{Q}}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\{$CM ~~types~~ types of $E\}$.

$$a_{(E,\Phi)}(\sigma) = |\Phi \wedge \sigma(\Phi)|$$

$$a^{0}_{(E,\Phi)}(\sigma) = \frac{1}{[G_{\mathbb{Q}} : \mathrm{Stab}(\Phi)]} \sum_{\overset{\tau}{\cancel{\Phi}} \in G_{\mathbb{Q}}/\mathrm{Stab}(\Phi)} a_{(E,\tau(\Phi))}(\sigma) .$$

Decompose into Artin characters

$$a^{0}_{(E,\Phi)} = \sum_{\chi} m(\chi)\, \chi$$

Define

Artin char.

$$h^{col}(E,\Phi) = \sum_{\chi} m(\chi) \left[\frac{L'(0,\chi)}{L(0,\chi)} + \frac{1}{2} \log (f_{\chi}) \right.$$

Conj. (Colmez): $h^{Falt}(E,\Phi) = h^{col}(E,\Phi)$.

(Certain periods compute certain logarithmic derivatives of L-functions.)

- If E is quadratic imaginary this is the Chowla-Selberg formula.
- If $E/\mathbb{Q}$ is abelian this is given by Colmez in the paper the conjecture is made up to removing an error term, which was later done by Obus.
- For some non-Galois quartic E, proved by T. Yang.
- Ongoing work of Bruinier-H.-Kudla-Rapoport-Yang: Any E containing a quadratic imaginary subfield (+ restriction on CM type).

Thm in progress (AGHMP) E any CM field

$$\sum_{\Phi} h^{Falt}(E,\Phi) \doteq \sum_{\Phi} h^{col}(E,\Phi).$$

$\doteq$ means equality holds up to a $\mathbb{Q}$-linear combinations of

$$\{\log p \mid p \text{ divides } \underbrace{2\,\mathrm{disc}(E)}_{\Delta}\}.$$

Let $\Phi_1, \ldots, \Phi_r$ be reps. for $G_{\mathbb{Q}}$-orbits of $\{\text{CM types}\}$.

Each (E, Φ_i) has dual $(E_i^{\#}, \Phi_i^{\#})$

Total reflex algebra

$$E^{\#} = \prod_i E_i^{\#}. \quad \text{has dim } 2^n.$$

$$\Phi^{\#} = \coprod_i \Phi_i^{\#} \subseteq \coprod \mathrm{Hom}(E_i^{\#}, \mathbb{C}) = \mathrm{Hom}(E^{\#}, \mathbb{C}).$$

Fact: $a^{\circ}_{(E^{\#},\Phi^{\#})} = \frac{1}{[E:\mathbb{Q}]} \sum_{\Phi} a^{\circ}_{(E,\Phi)}$.

$$\begin{array}{ccc} h^{Falt}(E^{\#},\Phi^{\#}) & \overset{?}{=} & h^{col}(E^{\#},\Phi^{\#}) \\ \shortparallel & & \shortparallel \\ \frac{1}{[E:\mathbb{Q}]} \sum_{\Phi} h^{Falt}(E,\Phi) & & \frac{1}{[E:\mathbb{Q}]} \sum_{\Phi} h^{col}(E,\Phi). \end{array}$$

Orthogonal Shimura Varieties:

Fix $\xi \in (E^{+})^{\times}$ neg. at exactly one ∞ place of E^{+}.

$(V,Q) = (E, \mathrm{Tr}_{E^{+}/\mathbb{Q}}\, \xi x \bar{x})$ has signature $(2n-2, 2)$.

Clifford algebra $C(V) = \left(\bigoplus_{k=0}^{\infty} V^{\otimes k}\right) / \langle v\otimes v - Q(v)\rangle$

$$\begin{array}{c} 1 \to \mathbb{G}_m \to GSpin(V) \to SO(V) \to 1 \\ \shortparallel \\ G \subset C(V)^{\times} \end{array}$$

$$\begin{array}{c} D = \{ z \in V_{\mathbb{C}} : [z,z] = 0,\ [z,\bar{z}] < 0 \} / \mathbb{C}^{\times} \\ \cap \\ \mathbb{P}(V_{\mathbb{C}}) \end{array}$$

Shimura data (G, D)

There is a canonical embedding $E^{\#} \to C(V)$.

$$\begin{tikzcd}
T = \{x \in E^\times \mid x\bar{x} \in \mathbb{Q}^\times\} \arrow[r] \arrow[d] & & G \arrow[d] \\
E^\times \arrow[r, "\text{reflex}", "\text{norm}"'] & (E^\#)^\times \arrow[r] & C(V)^\times
\end{tikzcd}$$

$T_E(\mathbb{R})$ acts on D with fixed points $\{z_0^+, z_0^-\}$

morphism $(T_E, \{z_0^+\}) \to (G, D)$

level structure

$$\begin{tikzcd}
\text{0-dim} \arrow[r] & Y_E(\mathbb{C}) = T_E(\mathbb{Q}) \backslash \{z_0^+\} \times T_E(\mathbb{A}_f) / K_E \arrow[d] \\
\text{dim } 2n-2 & M(\mathbb{C}) = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f) / K
\end{tikzcd}$$

Canonical models over E.

Kuga - Satake abelian scheme

G acts on $C(V)$ by left mult.

$G \to GSp(C(V))$ for some symplectic form.

$\rightsquigarrow$ $M \hookrightarrow$ Siegel moduli space

$\rightsquigarrow$ A dim 2^{2n-1} has right action of $C(V)$

$$\begin{tikzcd}
A \arrow[d] \\
M
\end{tikzcd}$$

$A|_{Y_E}$ has action of $E^\# \otimes C(V) \cong Mat_{2^n}(E^\#)$.

Prop. $\exists$ abelian scheme $\begin{tikzcd} B \arrow[d] \\ Y_E \end{tikzcd}$ with CM by $\mathcal{O}_{E^\#}$.

and type $\Phi^{\#}$ and ~~[illegible]~~ Δ-isogeny.

$$A|_{Y_E} \longrightarrow \underbrace{B\times\cdots\times B}_{2^n \text{ times}}$$

Divisors $Z(m)$ on M

local structure of $\mathbb{Q}$-vector spaces $H_1(A,\mathbb{Q})$ on $M(\mathbb{C})$ defined by $G \to GSp(C(V))$.

local system $\mathbb{V}$ defined by $G \to SO(V)$

$$V \hookrightarrow C(V) \xrightarrow[\text{mult}]{\text{left}} End(C(V))$$

inclusion of local systems $\mathbb{V} \to End(H_1(A_z,\mathbb{Q}))$.

Def: Given $s \in M(\mathbb{C})$ an endo. x of A_s is special if $H_1(x) \in \mathbb{V}_s$.

Given* a connected scheme $S \to M$ and $x \in End(A_S)$, x is special if special at one (any) complex point $s \in S$.

$V(A_S) = \{$special endomorphisms$\}$ is pos. def. quadratic space via $Q(x) = x \circ x \in \mathbb{Z}$.

$Z(m)$ has S-points $\{x \in V(A_S) : Q(x) = m\}$.

$$\begin{tikzcd} Z(m) \arrow[d, "\text{Codim}=1"] \\ M \end{tikzcd}$$

Kisin & Vasiu

integral models over $\mathcal{O}_E[1/\Delta]$

$$\begin{tikzcd}
 & \mathcal{A} \arrow[d] & \\
\mathcal{Z}(m) \arrow[r] & \mathcal{M} & \mathcal{Y}_E \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{\mathrm{Pic}}(\mathcal{M}) \arrow[r] & \widehat{\mathrm{Pic}}(\overset{\swarrow \text{1-dim}}{\mathcal{Y}_E}) \arrow[r, "\widehat{\deg}"] & \mathbb{R} \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\
\det(\pi_*\Omega^1_{\mathcal{A}/\mathcal{M}}) \arrow[r, mapsto] & \det(\pi_*\Omega^1_{\mathcal{B}/\mathcal{Y}_E}) \arrow[r, mapsto] & h^{Falt}(E^\#, \Phi^\#)
\end{tikzcd}$$

<u>Borcherds Products:</u>

Fix $f(\tau) = \sum_{m \gg -\infty} c_f(m) q^m \in M^!_{2-n}(SL_2(\mathbb{Z}), \mathbb{Z})$

Borcherds constructs a rational section

$\Phi(f)$ of $\det(\pi_* \Omega^1_{\mathcal{A}/\mathcal{M}})^{\otimes c_f(0)}$

with $\mathrm{div}(\Phi(f)) = \sum_{m>0} c_f(-m) Z(m)$.

Thus, $c_f(0) \cdot h^{Falt}(E^\#, \Phi^\#)$

$$\rotatebox{90}{$=$}$$

$$I(\mathrm{div}\,\Phi(f), \mathcal{Y}_E) - \sum_{\sigma : E \to \mathbb{C}} \sum_{y \in \mathcal{Y}_E^\sigma(\mathbb{C})} \log \| \Phi(f)^\sigma(y) \|$$

$\nwarrow$ intersection mult. on $\mathcal{M}$

BKY construct Hilbert modular Eisenstein series

$G_E(\tau, s)$, $\tau \in \mathcal{H} \times \cdots \times \mathcal{H}$.

Howard
12-2-14

$$G'_E(\tau,0) = \sum_{\substack{\alpha \in E^+ \\ \alpha \gg 0}} a_E(\alpha) q^{\alpha} + a_E(0) + \log|\mathrm{Im}(\tau)|$$

other terms include all involving $\mathrm{Im}(\tau)$.

Formal q-exp

$$\mathcal{G}_E(\tau) = a_E(0) + \sum_{\alpha \gg 0} a_E(\alpha) q^{\alpha}$$

diagonal restriction gives

$$\mathcal{G}(\tau) = \sum_{m \gg 0} a(m) q^m$$

Thm (BKY):

$$-\sum_{\sigma} \sum_{y} \log\|\Psi(f)(y)\| = \sum_{m > 0} a(m)\, c_f(-m)$$

Thm (AGHMP):

$$I(\mathcal{Z}(m), \mathcal{Y}_E) = {}^{-}a(m)$$

so

$$I(\dim \Psi(f), \mathcal{Y}_E) = -\sum_{m>0} a(m)\, c_f(-m).$$

So $\quad c_f(0) \cdot h^{\mathrm{Falt}}(E^{\#}, \Phi^{\#}) = a(0)\, c_f(0).$

$$h^{\mathrm{Falt}}(E^{\#}, \Phi^{\#}) = a(0) = \frac{L'(0, \chi_{E/E^{\#}})}{L(0, \chi_{E/E^{\#}})} = h^{\mathrm{col}}(E^{\#}, \Phi^{\#}).$$