Vignéras

12-1-14

PS1

Parabolically induced mod p representations of reductive p-adic

Ho - Henniart - Herzig - V.

$F/\mathbb{Q}_p < \infty$

$\mathbb{F}_p((t))$

$\underline{G}$ reductive connected group over F

$G = \underline{G}(F)$, $T < B = Z\,U$ min. parabolic

C alg. closed field of char. p.

Representations admissible π of G.

(π^H has fin. dim. for $H < G$ open compact.)

$\mathrm{Irr}_C^{adm}(G)$ — supercuspidal / non-supercuspidal

Classify the non-supercuspidal irr. adm. reps.

Barthel-Livné : $GL(2,F)$

Herzig : $GL(n,F)$ all n, $F \supset \mathbb{Q}_p$

Abe : split $F \supset \mathbb{Q}_p$

A.H.H.V: general G and F

$P \supset B$ $\quad \mathrm{Ind}_P^G : \underbrace{\mathrm{Mod}_C M}_{\text{rep. of } M/C} \longrightarrow \mathrm{Mod}_C(G)$

$\shortparallel$

MN

$\cup$

Z

(τ, W) $\qquad \mathrm{Ind}_P^G(W)$ ↻ G right translation

$= \{ f : G \to W : f(nmgk) = \tau(m) f(g) \}$

K open compact $< H$

Prop.: Parabolic induction is faithful, with a left and right adjoint, respects admissibility.

Def: $\pi \in \mathrm{Irr}^{adm}(G)$ is supercuspidal if it is not a subquotient of $\mathrm{Ind}_P^G \tau$ for $P \neq G$, $\tau \in \mathrm{Irr}^{adm}(M)$.

supercuspidal admissible triple: (P, σ, Q) $\quad B \subset P = MN$
$\sigma \in \mathrm{Irr}_C^{adm}(\pi)$ supercuspidal, $P \subset Q \subset P(\sigma)$ where $P(\sigma)$ is the largest group where σ can be extended trivially on N.

Generalized Steinberg representation: $St_Q^{P(\sigma)}(\sigma) :=$

$$\frac{\mathrm{Ind}_Q^{P(\sigma)} e(\sigma)}{\sum_{Q \subsetneq Q' \leq P(\sigma)} e(\sigma)}$$

$$I(P, \sigma, Q) = \mathrm{Ind}_{P(\sigma)}^G St_Q^{P(\sigma)}(\sigma).$$

Thm: There is a bijection

$$(P, \sigma, Q) \longrightarrow I(P, \sigma, Q)$$
$$\{\text{supercusp. adm. triple}\} \longrightarrow \mathrm{Irr}_C^{adm}(G).$$

First applications ① $\underbrace{\mathrm{Ind}_P^G}\tau \qquad \tau \in \mathrm{Irr}_C^{adm}(M)$

finite length, irred subqts are adm, mult. 1.

② supercuspidal support

Second applications:

Thm: supercuspidal = supersingular.

Satake homomorphism:

K open compact subgroup. "special parahoric" $G = BK$

$B \cap K = Z_0 U_0$ $\quad P \cap K = M_0 N_0$

$C[K \backslash G / K]$, $*$ convolution product

spherical Hecke algebra

$$\begin{tikzcd} C[K\backslash G/K] \arrow[d, hook] \arrow[dd, bend left, "\simeq"] \\ C[Z/Z_0], * \\ C[Z^+/Z_0], * = \text{image}. \end{tikzcd}$$

$V \in \mathrm{Irr}_C(K)$. compact induction $\mathrm{ind}_K^G V$

$= \{ f : G \to V$ compact support

$f(kg) = k f(g) \}$

Generalization of spherical Hecke algebra:

$\mathrm{End}_{CG}(\mathrm{ind}_K^G V) = \mathcal{H}(G, K, V)$

$\qquad = \{ \underset{\text{compact support}}{f : G \to \mathrm{End}_C(V)}, f(k_1 g k_2) = k_1 f(g) k_2 \}$

$$\begin{tikzcd} \mathcal{H}(G,K,V) \arrow[r, hook] \arrow[dr, "\simeq"] & \mathcal{H}(Z, Z_0, \underbrace{V_{U_0}}_{\psi_V}) \\ & \mathcal{H}(Z^+, Z_0, \psi_V) \end{tikzcd}$$

$Z_0 \circlearrowright V_{U_0}$ has dim 1

Z_0 ψ_V a char.

Vigneras
12-1-17
p94

Z/Z_0 is commutative (Herzig)

Z/Z_u not comm. in general
proper Sylow.

center $\mathcal{Z}(Z^+, Z_0, \Psi_V)$.

$$\chi \otimes_{\mathcal{Z}(Z^+, Z_0, \Psi_V)} \operatorname{ind}_K^G V \qquad \chi : \mathcal{Z}(Z^+, Z_0, \Psi_V) \longrightarrow C$$
character

$\pi \in \mathrm{Irr}_C^{adm}(G)$

<u>Def:</u> V is a weight of π, χ is an eigenvalue of $\mathcal{Z}(G, K, V)$ in π if π is a quotient of $\chi \otimes \operatorname{ind}_K^G V$

$$\begin{tikzcd}
\mathcal{Z}(G, K, V) \arrow[rr] \arrow[dr] & & \mathcal{Z}(Z, Z_0, V_{U_0}) \\
& \mathcal{H}(M, M_0, V_{N_0}) \arrow[ur] &
\end{tikzcd}$$
irred. rep. of M_0.

χ is called supersingular if it does not extend to $\mathcal{Z}(M, M_0, V_{N_0})$ for all $P = MN_0 \neq G$

π is <u>supersingular</u> if for all weights V in π, all eigenvalues χ in π of χ is supersingular.

Going from compact induction to parabolic induction
$$C[K\backslash G] \hookrightarrow \mathrm{Ind}_B^G C[Z_0 \backslash Z]$$ equivalent to $\mathcal{H}(G, K, V)$.

$$\mathrm{ind}_K^G V \hookrightarrow \mathrm{Ind}_B^G\ \mathrm{ind}_{Z_0}^Z V_{U_0}$$
$$\tilde{\Psi}_V$$

V "regular" $\hookrightarrow \mathrm{Ind}_P^G\ \mathrm{ind}_{M_0}^M V_{N_0}$ $\quad \chi \in \tilde{Z}(M, M_0, V_{N_0})$.

restriction of char of $Z(G, K, V)$

$$\chi \otimes \mathrm{ind}_K^G \simeq \chi \otimes \mathrm{Ind}_P^G\ \mathrm{ind}_{M_0}^M V_{N_0}.$$

<u>change of weight</u> under some condition

$$\chi \otimes \mathrm{ind}_K^G V \simeq \chi \otimes \mathrm{ind}_K^G V' \qquad \Psi_V = \Psi_{V'}.$$
$$Z(G, K, V') \simeq Z(Z^+, Z_0, \Psi_V)$$

proved by using the action of the pro-p-Iwahori

Compare $\mathrm{ind}_K^G V$ and $\mathrm{ind}_K^G V'$ inside of

$\mathcal{H}_c(G, I_u)$ $\quad I_u$ = pro-p-radical of the Iwahori I

$$I_u \cap B_0 = Z_u U_0.$$

$$\mathrm{Mod}_c(G) \xrightarrow{I_u\text{-inv}} \mathrm{Mod}_{\mathcal{H}_c}^{f}(G, I_u).$$

classification of

left adjoint $- \otimes C[I_u \backslash G]$. $\qquad \mathrm{Irr}_{\mathcal{H}_c(G, I_u)}$

$V \in \mathrm{Rod}_c G \Rightarrow V^{I_u} \neq 0.$ $\qquad$ supersingular / non-supersingular

$\neq 0$

Vigneras
12-1-14.
p. 6

$$\begin{tikzcd}
\mathrm{Mod}_C(G) \arrow[r, shift left, "I_u\text{-inv}"] & \mathrm{Mod}_{\mathcal{H}_C(G, I_u)} \arrow[l, shift left] \\
\mathrm{Mod}_C(M) \arrow[u, "\mathrm{Ind}_P^G"] \arrow[r, shift left, "?\text{-inv}"] & \mathrm{Mod}_{\mathcal{H}_C}(M, I_u, ?) \arrow[u, "\mathrm{Ind}_P^G"'] \arrow[l, shift left]
\end{tikzcd}$$

<u>Thm:</u> 1) Parabolic induction commutes with pro-p-invariant functor and its left adjoint.

2) $\pi \in \mathrm{Irr}_C^{adm}(G)$ supersingular $\iff$ π^{I_u} has all its irreducible subquotients supersingular.