Mock modular forms and their shadows

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Basic Definitions

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Notation: Throughout, let

$$z = x + iy \in \mathbb{H} \quad \text{with} \quad x, y \in \mathbb{R} \quad \text{and} \quad y > 0.$$

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2. *We have that f is holomorphic at the cusps.*

Fourier expansions

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Lemma

If f is a weight k modular form, then

$$f(z) = \sum_{n \geq 0} c_f(n) q^n \quad \text{where} \quad q := \exp(2\pi i z).$$

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- $S_k :=$ weight k **cusp forms**
  (subspace of M_k whose forms have vanishing constant terms)

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Definition

If $f \in S_k$ and $0 \leq n \leq k-2$, then the nth **period** of f is

$$r_n(f) := \int_0^\infty f(it)t^n dt.$$

Periods and L-values

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Remark

For these n, we have the following relation with critical values

$$L(f, n+1) = \frac{(2\pi)^{n+1}}{n!} \cdot r_n(f).$$

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then

$$r(f;z) = \sum_{n=0}^{k-2} i^{-n+1} \binom{k-2}{n} \cdot r_n(f) \cdot z^{k-2-n}.$$

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If $P \in \mathbf{V} := \mathbf{V}_{k-2}(\mathbb{C})$ = polynomials of degree $\leq k-2$, and $\gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then let

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Let $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $U := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, *and let*

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Then $\mathbf{W}$ *is the cohomology group* $H^1(\mathrm{SL}_2(\mathbb{Z}), \mathbf{V})$.

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Theorem (Eichler-Shimura)

If $\mathbf{W}_0 \subset \mathbf{W}$ *is the codim. 1 space not containing* $z^{k-2} - 1$, *then*

$$r : S_k \longrightarrow \mathbf{W}_0$$

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A larger theory?

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Question

Is this all part of a larger theory?

Harmonic Maass Forms

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- Partitions and q-series identities
- Moonshine for affine Lie superalgebras
- Borcherds' products
- Donaldson invariants (Moore-Witten Conjecture)

Basic Definitions

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Hyperbolic Laplacian:

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

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Notation

The space of weight k harmonic Maass forms is denoted H_k.

Fourier expansions

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Lemma (Bruinier, Funke)

If $F \in H_{2-k}$ and $\Gamma(a, x)$ is the incomplete Γ-function, then

$$F(z) = \sum_{n \gg -\infty} c_F^+(n) q^n + \sum_{n<0} c_F^-(n) \Gamma(k-1, 4\pi|n|y) q^n.$$

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Holomorphic part F^+ **Nonholomorphic part** F^-

Remark

The function F^+ is called a **mock modular form**.

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- $D^{k-1} : M_{2-k}^! \longrightarrow S_k^!$ (Bol),
- $D^{k-1} : H_{2-k} \longrightarrow S_k^!$ (Bruinier, Ono, Rhoades).

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Lemma (Bruinier, Funke)

If $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \overline{z}}}$, *then*

$$\xi_{2-k} : H_{2-k} \twoheadrightarrow S_k.$$

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Remark

The cusp form $\xi_{2-k}(F)$ *is called the* **shadow** *of* F^+.

Period polynomials revisited

Definition

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Remark

For $f \in S_k^!$, the above integral may be divergent.

The Regularized integral

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Consider a continuous function $f : \mathbb{H} \to \mathbb{C}$ and

$$f(z) = O(e^{c \operatorname{Im} z})$$

for some $c \in \mathbb{R}^{+}$ as $\operatorname{Im} z \to \infty$.

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Definition (Fricke, Rankin-Selberg)

For $t \gg 0$, if

$$\int_i^{i\infty} e^{itz} f(z)\, dz$$

has an analytic continuation to $t = 0$, then we define

$$R.\int_i^{i\infty} f(z)\, dz := \left[\int_i^{i\infty} e^{itz} f(z)\, dz\right]_{t=0}.$$

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Definition

For $f \in S_k^!$, define

$$r(f; z) := R. \int_0^{i\infty} f(\tau)(z-\tau)^{k-2} d\tau.$$

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Recall the extended Bol-type identity:

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Remark

Mock modular forms "are" regularized iterated integrals of weakly holomorphic cusp forms.

Mock modular periods generate critical L-values

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Theorem (Bringmann, Guerzhoy, Kent, Ono)

If $F \in H_{2-k}$ and $g = \xi_{2-k}(F) \in S_k$, then

$$F^+(z) - z^{k-2}F^+(-1/z) = \sum_{n=0}^{k-2} (-1)^n \frac{\overline{L(g, n+1)}}{(k-2-n)!} \cdot (2\pi i z)^{k-2-n}.$$

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New Eichler-Shimura isomorphisms

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The following diagram is commutative:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & S_k \oplus S_k & \xrightarrow{r^- + ir^+} & \mathbf{W}_0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & D^{k-1}(S^!_{2-k}) & \longrightarrow & S^!_k & \xrightarrow{r} & \mathbf{W} & \longrightarrow & 0 \\
 & & \uparrow {\scriptstyle =} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & D^{k-1}(S^!_{2-k}) & \longrightarrow & D^{k-1}(M^!_{2-k}) & \xrightarrow{r} & \langle z^{k-2} - 1 \rangle & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

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The isomorphism

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suggests that there are more eigenforms than just those in S_k.

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Definition

For any positive integer $m \geq 2$, let $T(m)$ be the usual weight k index m Hecke operator.

More eigenforms

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Definition

We say $f \in S_k^!$ is a *Hecke eigenform* if for every Hecke operator $T(m)$ there is a complex number λ_m for which

$$(f \mid_k T(m) - \lambda_m f)(z) \in D^{k-1}\left(M_{2-k}^!\right).$$

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Remark

This extends the usual definition of Hecke eigenform for S_k.

Multiplicity two theorem

Theorem (Bringmann, Guerzhoy, Kent, Ono)

Let $d = \dim S_k$. *Then*

$$S_k^!/D^{k-1}(M_{2-k}^!) = \bigoplus_{i=1}^{d} \mathbb{T}_i$$

where each $\mathbb{T}_i$ *is a 2-dimensional Hecke eigenspace.*