

L-functions, p-adic L-fns, and rational points:

(BSD)  $E$  ell. curve/ $\mathbb{Q}$  :  $\text{rank } E(\mathbb{Q}) = \text{ord}_{s=1} L(E, s) =$   
 "local-global" principle

Known: When analytic rank  $\leq 2$

Thm (Hass-Zagier): When analytic rank = 2  $\Rightarrow$  alg. rank  $\geq 2$

Uses an auxiliary imag. quad. field  $K \rightsquigarrow P_K \in E(K)$

$$L_K(E, s) = L(E, s) L(E, \varepsilon_K, s)$$

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(Choose  $K$  so SSA of  $L_K(E, s) = -1$ )

Formula:

$$\begin{aligned} L'_K(E, s) \Big|_{s=1} &= \langle P_K, P_K \rangle_{NT} \\ \parallel & \\ (L(E, s) L(E, \varepsilon_K, s))' \Big|_{s=1} & \\ \parallel & \text{(assume signs above + +)} \\ L'(E, 1) L(E, \varepsilon_K, 1) & \\ \neq 0 & \\ \Rightarrow P_K \in E(\mathbb{Q}) & \end{aligned}$$

up to known factors

$H =$  Hilbert class field of  $K$

$$\chi: \text{Gal}(H/K) \rightarrow \mathbb{C}^\times$$

$$\Theta_\chi = \sum_{\alpha \in \mathcal{O}_K} \chi(\alpha) e^{2\pi i (\text{Nm}(\alpha)) z}$$

modulus form of wt 2.

$$L'(E \otimes \Theta_\chi, s) \Big|_{s=1} = \langle P^\chi, P^\chi \rangle$$

$$P^\chi \in (E(H) \otimes \mathbb{C})^\chi \quad \chi=2 \quad \bar{\phantom{x}}$$

Generalizations:

(a) Zhang :  $f$  mod. form of wt  $k$ . (even  $k$ )  $\text{Center} = k/2$   
 $K, \chi: \text{Gal}(H/K) \rightarrow \mathbb{C}^\times$

$$L(f \otimes \chi, s)$$

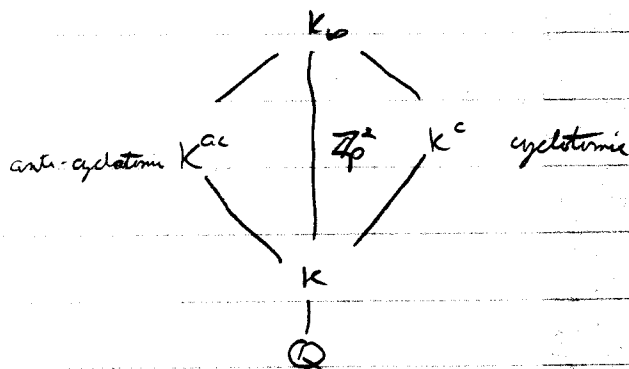
$$s = k/2$$

"Heegner hypothesis"  $\Leftrightarrow s, n = -1$ .

$$L'(f \otimes \chi, s) \longleftrightarrow \text{Beilinson-Bloch}$$

Zhang constructed "Heegner cycles"

(b) Perrin-Riou :  $p$ -adic direction  
 $E/\mathbb{Q}$



$p$ -adic  $L$ -function

$$L(E, \chi, -) \xrightarrow{\text{interpolates } \chi \text{ of finite order } \mathcal{L}_1}$$

2  $p$ -adic  $L$ -functions  $\rightarrow$  interpolates "Heegner classes" in a certain range.  $\mathcal{L}_2$

Thm (Perrin-Rivin):  $\mathcal{L}_1^{(\text{cycl})} \longleftrightarrow \langle P_K, P_K \rangle_{p\text{-adic}}$

(it vanishes on anti-cycl. part) This L-fun has 2 variables, one for each  $\mathbb{Z}_p$ -ext.

Question: What if  $\chi$  is a Hecke charact of  $K$ ?

$$\chi: K_{\mathbb{A}}^{\times} \rightarrow \mathbb{C}^{\times} \quad \chi \text{ has wt } d. \quad \chi|_{\mathbb{A}_{\mathbb{Z}}^{\times}} = \text{Nm}^d \chi_x$$

$$\chi(x x_{\infty}) = \chi(x) \chi_{\infty}^{-d}$$

Ex:  $K$  class # 1  $K = \mathbb{Q}(\sqrt{-D})$ ,  $D$  odd = disc

$$(\mathcal{O}_K / \sqrt{-D} \mathcal{O}_K)^{\times} \cong (\mathbb{Z} / D\mathbb{Z})^{\times} \xrightarrow{\chi_K} \{\pm 1\}$$

$$\psi(\alpha) = \alpha \chi_K(\alpha) \quad \text{Hecke char. of wt } 1.$$

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f mod form of wt  $K$ , char.  $\chi_f$  on  $\Gamma_0(N)$ ,  $N$  sq. free for simplicity

$K$  imag. quad. field.

\* "Heegner hypothesis":  $K$  is split or ramified at all  $q|N$ .

$\chi$  Hecke char. of  $K$  and wt,  $d$

Assume  $\text{Norm}(\mathcal{O}_K^{\times}) \mid N$

\* Assume  $\chi_f \cdot \chi_x = 1$

crucial assumption

$$\Rightarrow K \hookrightarrow M_2(\mathbb{C}) \quad \text{s.t. } \mathcal{O} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

$$\mathcal{O} \cap K = \mathcal{O}_K$$

$$K^{\times} \hookrightarrow \text{GL}_2(\mathbb{C})^{\times} \hookrightarrow \text{GL}_2(\mathbb{R})^{\times}$$

Let  $z =$  unique fixed pt of  $K^*$  on  $S$ .

$$S \longrightarrow \mathcal{H} / \Gamma_0(N) = Y_0(N)$$

$$z \longmapsto [z] \in Y_0(N)(H)$$

$$[z] - [\infty] \xrightarrow{\text{comp at } \infty} \in \text{Div}(X_0(N)) \xrightarrow[\text{Abel-Jacobi map}]{AJ} J_0(N)(H)$$

(a)  $f$  has wt 2  $\iff E$  gives  $J_0(N)(H) \rightarrow E(H)$

Call the point in  $E(H)$  you get  $P_z$ .

$$P^x = \sum_{\sigma \in \text{Gal}(H/K)} X'(\sigma) P_z^\sigma \in (E(H) \otimes \mathbb{C})^x$$

$$\chi: \text{Gal}(H/K) \rightarrow \mathbb{C}^x$$

$$\chi = \mathbf{1}: E(H) \xrightarrow{+} E(K) \xrightarrow{P_K} \mathbb{C}$$

$$(L(E, s) L(E, \epsilon_K, s))' = \langle P_K, P_K \rangle$$

$$+ \quad - \quad \rightsquigarrow P_K \in E(K) \quad \text{b/c c.c. acts by } -1$$

$$- \quad + \quad \rightsquigarrow P_K \in E(\mathbb{Q})$$

This says the "point knows where to live."

(b) Zhang:

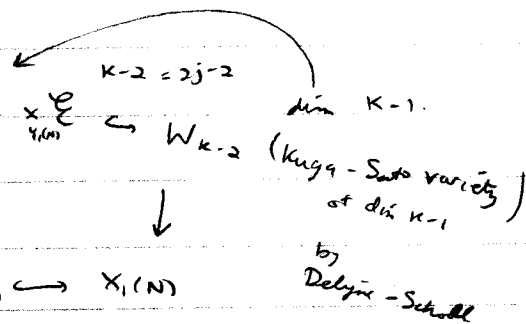
$f$  has wt  $k=2j$ ,  $\chi$  finite order.

$\Sigma$  = universal elliptic curve

$$Y_1(N)$$

$$\Sigma \times_{Y_1(N)}$$

$$Y_1(N) \hookrightarrow X_1(N)$$



Heegner pt [z] ...

Joint work by Bertolini & Darmon:

$f$  wt  $k$ ,  $x$  wt  $l$ .

(3a)  $l < k$

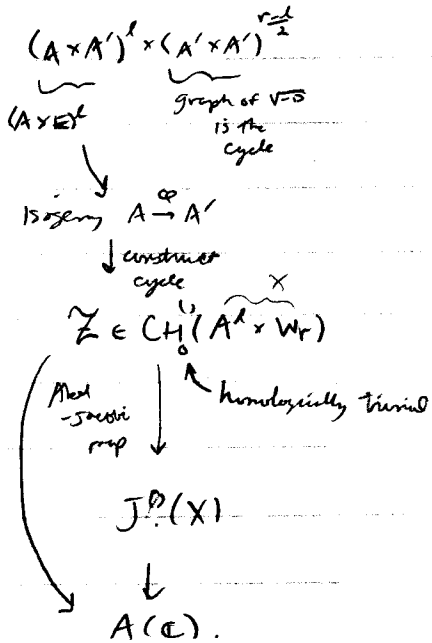
$\text{sgn} = -1$

Pair  $A$  a CM (by  $\mathcal{O}_K$ )  
elliptic curve.

$$r = k - 2$$

$$\begin{array}{ccc} A^l \times W_r & A^l \times (A')^r \\ \downarrow & \downarrow \\ X_1(N) & [z] \end{array}$$

$r \geq l$



$k \in l \pmod{2} \Rightarrow$  center is central.

(3b)  $l > k$

$\text{sgn} = +1$

$l = k$ :

$$K \hookrightarrow M_2(\mathbb{Q})$$

$$K^* \hookrightarrow GL_2(\mathbb{Q})$$

$$K_{/A}^* \hookrightarrow GL_{2,\mathbb{A}}(A)$$

$$\begin{array}{c} f \longrightarrow F \text{ fctn on } GL_2(A) \\ \downarrow \\ \mathbb{C} \end{array}$$

$$L_X(f) = \int F \cdot X = \text{"a finite sum of values of } f \text{ at CM points twisted by } X"$$

$\underbrace{\quad \quad \quad}_{\text{open compact}} \quad \underbrace{\quad \quad \quad}_{K_{/A}^* / K_{\infty}^*}$

Waldspurger formula:

$$L_X(f)^2 = L(f \otimes \theta_X, \text{center})$$

$l > k$ :  $l = k + 2t$

Use Shimura-Muro operator to bump up the wt of  $f$  (bump up by 2 each time)

$\psi$  char of wt 1 of  $K$

$$\chi = \psi^r$$

$$f = \Theta_{\psi^{rn}} \quad (\text{form } \eta \text{ wt } r+2)$$

$$L(f \otimes \Theta_r, s) = L(\psi^{rn}, s) L(\psi, s-r)$$

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