PANTS IX

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L-functions, p-adic L-fctns, and rational points:

(BSD) E ell. curve$/\mathbb{Q}$: $\underset{\text{algebraic}}{\operatorname{rank} E(\mathbb{Q})} = \underset{\text{analytic rank}}{\operatorname{ord}_{s=1} L(E,s)}$:

"local-global" principal

Known: When analytic rank ≤ 1

Thm (Gross-Zagier): When analytic rank $= 1 \Rightarrow$ alg. rank ≥ 1

Uses an auxiliary imag. quad. field $K \rightsquigarrow P_K \in E(K)$

$$L_{/K}(E,s) = \overbrace{L(E,s)\,L(E,\varepsilon_K,s)}^{\deg 4}$$

(Choose K so sgn of $L_{/K}(E,s) = -1$)

Formula: $$L'_{/K}(E,s)\big|_{s=1} \overset{\text{up to known factors}}{\doteq} \langle P_K, P_K\rangle_{NT}$$

$$\| $$

$$(L(E,s)\,L(E,\varepsilon_K,s))'_{s=1}$$

$$\| \qquad \text{(assume sgns above } - \ + \text{)}$$

$$L'(E,1)\,L(E,\varepsilon_K,1)$$

$$\neq 0$$

$\Rightarrow P_K \in E(\mathbb{Q})$.

H = Hilbert class field of K

$$\chi : \mathrm{Gal}(H/K) \longrightarrow \mathbb{C}^\times$$

$$\Theta_\chi = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \chi(\sigma)\, e^{2\pi i (N m \sigma) z} \qquad \text{modular form of wt 1.}$$

$$L'(E \otimes \Theta_\chi, s)_{s=1} = \langle P^\chi, P^\chi \rangle$$

$$P^\chi \in (E(H) \otimes \mathbb{C})^\chi \qquad \chi = 1 \text{ is}$$

Generalizations:

(a) Zhang : f mod. form of wt k. (even k) center $= k/2$

K, $\chi : Gal(H/K) \to \mathbb{C}^\times$

$$L(f \otimes \theta_\chi, s)$$

$$s = k/2$$

"Heegner hypothesis" $\Rightarrow sgn = -1$.

$$L'(f \otimes \theta_\chi, s) \longleftrightarrow \quad \text{Beilinson - Bloch}$$

Zhang constructed "Heegner cycles"

(b) Perrin-Riou : p-adic direction

$E/\mathbb{Q}$

$$\begin{tikzcd}
 & K_\infty \arrow[dl, dash] \arrow[dd, dash, "\mathbb{Z}_p^2"] \arrow[dr, dash] & \\
\text{anti-cyclotomic } K^{ac} \arrow[dr, dash] & & K^c \text{ cyclotomic} \arrow[dl, dash] \\
 & K \arrow[d, dash] & \\
 & \mathbb{Q} &
\end{tikzcd}$$

p-adic L-function

$L(E, \chi, -)$ — interpolates χ of finite order $\mathcal{L}_1$

2 p-adic L-functions — interpolates "Hecke chars" in a certain range. $\mathcal{L}_2$

Thm (Perrin-Riou): $\mathcal{L}_1^{(cycl)} \longleftrightarrow \langle P_K, P_K \rangle_{p\text{-adic ht.}}$

(it vanishes on anticycl. part) This L-fctn has 2 variables, one for each $\mathbb{Z}_p$-ext.

Question: What if χ is a Hecke character of K?

$$\chi : K_{/\mathbb{A}}^\times \longrightarrow \mathbb{C}^\times \qquad \chi \text{ has wt } \ell. \qquad \chi|_{\mathbb{A}_\mathbb{Q}^\times} = \mathrm{Nm}^\ell \, \varepsilon_\chi .$$

$$\chi(x x_\infty) = \chi(x) \, x_\infty^{-\ell}$$

Ex: K class # 1 $\quad K = \mathbb{Q}(\sqrt{-D})$, D odd $=$ disc

$$\left(\mathcal{O}_K / \sqrt{-D}\,\mathcal{O}_K\right)^\times \xrightarrow{\sim} \left(\mathbb{Z}/D\mathbb{Z}\right)^\times \xrightarrow{\varepsilon_K} \{\pm 1\}$$

$\Psi((\alpha)) = \alpha \, \varepsilon_K(\alpha)$ Hecke char. of wt 1.

$\doteq$

f mod form of wt k, char ε_f on $\Gamma_0(N)$, N sq. free for simplicity

K imag. quad. field.

* "Heegner hypothesis": K is split or ramified at all $q \mid N$.

χ Hecke char. of K and wt. ℓ

Assume $\mathrm{Norm}(c_\chi) \mid N$ "cond of χ"

* Assume $\varepsilon_f \cdot \varepsilon_\chi = 1$

crucial assumptions

$\Rightarrow K \hookrightarrow M_2(\mathcal{O})$ s.t. $\mathcal{O} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\}$

$\mathcal{O} \cap K = \mathcal{O}_K$

$K^\times \hookrightarrow GL_2(\mathbb{Q})^+ \hookrightarrow GL_2(\mathbb{R})^+$

Let τ = unique fixed pt of $K^\times$ on $\mathfrak{H}$.

$$\mathfrak{H} \longrightarrow \Gamma_0(N) \backslash \mathfrak{H} = Y_0(N)$$

$$\tau \longmapsto [\tau] \in Y_0(N)(H)$$

$$[\tau] - [\infty] \in \mathrm{Div}(X_0(N)) \xrightarrow[\text{Abel-Jacobi map}]{AJ} J_0(N)(H)$$

cusp at ∞

(a) f has wt 2 $\longleftrightarrow E$ gives $J_0(N)(H) \longrightarrow E(H)$

Call the point in $E(H)$ you get P_τ.

$$P^\chi = \sum_{\sigma \in Gal(H/K)} \chi^{-1}(\sigma)\, P_\tau^\sigma \in (E(H) \otimes \mathbb{C})^\chi$$

$$\chi : Gal(H/K) \to \mathbb{C}^\times$$

$$\chi = \mathbb{1} : \quad E(H) \xrightarrow{tr} \overset{P_K}{\overset{\rotatebox{90}{$\in$}}{E(K)}} \circlearrowleft c$$

$$(L(E,s)\, L(E,\varepsilon_K,s))' = \langle P_K, P_K \rangle$$

$+ \quad - \quad \rightsquigarrow P_K \in E(K)$ b/c c.c. acts by -1

$- \quad + \quad \rightsquigarrow P_K \in E(\mathbb{Q})$

This says the "point knows where to live."

(b) Zhang:

f has wt $k = 2j$, χ finite order

$$\begin{array}{ccc} \mathcal{E} & & \mathcal{E} \times_{Y_1(N)} \cdots \times_{Y_1(N)} \mathcal{E} \hookrightarrow W_{k-2} \\ \downarrow & & \downarrow \qquad\qquad \downarrow \\ Y_1(N) & & Y_1(N) \hookrightarrow X_1(N) \end{array}$$

$\mathcal{E}$ = universal elliptic curve

$k-2 = 2j-2$

dim $k-1$.

(Kuga-Sato variety of dim $k-1$)

by Deligne-Scholl

Heegner pt $[\tau]$ . . .

<u>Joint work by Bertolini & Darmon:</u>

f wt k, χ wt ℓ.

$k \equiv \ell \pmod 2 \Rightarrow$ center is critical.

(3a) $\ell < k$

sgn $= -1$

Pick A a CM (by $\mathcal{O}_K$) elliptic curve.

$r = k-2$

$$\begin{array}{ccc} A^\ell \times W_r & & A^\ell \times (A')^r \\ \downarrow & & \downarrow \\ X_1(N) & & [\tau] \end{array}$$

$r \geq \ell$

$$\underbrace{(A \times A')^\ell}_{(A\times E)^\ell} \times \underbrace{(A' \times A')}_{\text{graph of } \sqrt{-D} \text{ is the cycle}}{}^{\frac{r-\ell}{2}}$$

$$\begin{array}{c} \downarrow \\ \text{Isogeny } A \xrightarrow{\varphi} A' \\ \downarrow \text{ construct cycle} \\ Z \in CH_0^{\ell}(\overbrace{A^\ell \times W_r}^{X}) \quad \text{homologically trivial} \\ \downarrow \text{ Abel-Jacobi map} \\ J^{?}(X) \\ \downarrow \\ A(\mathbb{C}). \end{array}$$

(3b) $\ell \geq k$

sgn $= +1$

$\ell = k$:

$K \hookrightarrow M_2(\mathbb{Q})$

$K^\times \hookrightarrow GL_2(\mathbb{Q})$

$K^\times_{/\mathbb{A}} \hookrightarrow GL_{2,\mathbb{Q}}(\mathbb{A})$

$$\begin{array}{c} f \longrightarrow F \text{ fcn on } GL_2(\mathbb{A}) \\ \downarrow \\ \mathbb{C}. \end{array}$$

$$L_\chi(f) = \int_{\text{open compact} \backslash K^\times_{\mathbb{A}} / K^\times_\infty} F \cdot \chi$$ = "a finite sum of values of f at CM points twisted by χ"

Waldspurger formula:

$$L_\chi(f)^2 \doteq L(f \otimes \theta_\chi, \text{center})$$

$\ell > k$: $\ell = k + 2t$

Use Shimura-Maass operator to bump up the wt of f (goes up by 2 each time)

ψ char of wt 1 of K

$\chi = \psi^r$

$f = \Theta_{\psi^{r+1}}$ (form of wt r+2)

$$L(f \otimes \Theta_\chi, s) = L(\psi^{2r+1}, s)\, L(\psi, s-r)$$

$$\begin{array}{ccc} - & + & - \\ & - & + \end{array}$$