An Euler system for automorphic symplectic motives:

Let M be a motive over F with coefficients in C. F a totally real number field, C a number field, and we fix an embedding $C \hookrightarrow \mathbb{C}$. We assume M is pure, weight -1, irreducible, $\exists$ $M \otimes M \longrightarrow C(1)$ symplectic, and automorphic (i.e. $\exists$ $\phi : \mathcal{L}_F \longrightarrow Sp_{2m}(\mathbb{C}) \simeq Sp(X) = \hat{G}$ s.t. $\forall$ place v of F,

$\mathcal{L}_F$ = Langlands group

$\phi_v = \phi|_{W_v} \simeq M_v(-\tfrac{1}{2})$)

W_v: Weil-Deligne group

- $\Pi(\phi) = \{(G,\pi) : \pi \in \Pi(G,\phi)\}/\sim$

$\subset \{(G,\pi) : G$ form of $SO(2m+1)/F$, π unitary complex rep. of $G(\mathbb{A}_F)$ s.t. $m(\pi) > 0\}/\sim$.

$m(\pi) = \dim \operatorname{Hom}_{G(\mathbb{A}_F)}(\pi, L^2_d(G(F)\backslash G(\mathbb{A}_F)))$.

$\pi = \otimes' \pi_v \in \Pi(G,\phi) \iff \forall v, \pi_v \in \Pi(G_v, \phi_v)$.

$$\{\pm 1\} \leq Z = \operatorname{Aut}_{\hat{G}}(\phi) \subset Z_v = \operatorname{Center}_{\hat{G}}(\phi_v).$$
$$A = \pi_0(Z) \longrightarrow A_v = \pi_0(Z_v).$$

<u>Conj. (Langlands - Arthur, ...):</u>

$$\Pi(\phi) \simeq \left(\prod_v A_v / \Delta A\right)^\vee \qquad (m(\pi) = 1)$$

$$(G, \pi = \otimes' \pi_v) \longleftrightarrow (c = (c_v))$$

$$\Pi(\phi_v) \longleftrightarrow A_v^\vee$$

$$(G_v, \pi_v) \longleftrightarrow c_v.$$

$$G = SO(V,\varphi) \qquad \mathrm{Witt}_v(V,\varphi) = c_v(-1)$$

E/F CM quadratic ext.

$$E[\infty] = (E^{ab})^{Ver(Gal_F^{ab})} = \bigcup_c E[c]$$

$E[c] =$ ring class field of conductor $c \subset \mathcal{O}_F$.

$\Gamma = Gal(E[\infty]/E)$

$\chi \in \Gamma^\vee$.

Then we can consider

$$\mathrm{Ind}\, \chi$$

a 2-dim. rep. of Gal_F, which is orthogonal. The L-functions of interest are

$$L(M,\chi,s) = L(\phi \otimes \mathrm{Ind}\,\chi, s+1/2)$$
$$= \prod_v L_v(\phi_v \otimes \mathrm{Ind}\,\chi_v, s+1/2).$$

$\rightsquigarrow$ $\varepsilon(M,\chi) = \varepsilon(\phi \otimes \mathrm{Ind}\,\chi) = \prod_v \varepsilon_v(\phi_v \otimes \mathrm{Ind}\,\chi_v)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cap$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \{\pm 1\}$.

- $\sigma \in \mathrm{Center}(Z_v)[2]$ (element of order 2).

$X(\sigma) = \{x \in X : \sigma x = -x\}$.

(*) $C_v(\chi,\sigma) = \varepsilon_v(X(\sigma) \otimes \mathrm{Ind}\,\chi_v) \times \eta_v(-1)^{\frac{1}{2}\dim X(\sigma)}$

$\eta = \otimes \eta_v : \hat{F}^\times / F^\times N\hat{E}^\times \longrightarrow \{\pm 1\}$

<u>Fact:</u> (1) (*) defines a character $c_v(\chi) \in A_v^\vee$

(2) $c_v(\chi) \equiv 1$ if ϕ_v NR. or $E_v \simeq F_v \times F_v$

(3) $c_v(\chi) = c_v(E)$ if $Ram(\chi) \cap Ram(M) = \phi$.

$$\varepsilon(M,\chi) = \prod_v c_v(\chi)(-1)$$

If $\varepsilon(M,\chi) = 1 \rightsquigarrow G = SO(V,\varphi)$ and π on G.

Lemma: $\exists\ (W, \psi)$ E-hermitian space s.t.

$$\underset{2n+1}{(V,\varphi)} \simeq \underset{2n}{(W, \mathrm{Tr}\,\psi)} \perp \underset{1}{(F_v, \varphi|F_v)}$$

$\rightsquigarrow \exists\ H = U(W, \psi) \subset G = SO(V, \varphi)$.

Conj: $L(M, \chi, 0) \neq 0 \iff f \longmapsto \int_{H(F)\backslash H(\mathbb{A})} f(h)\bar{\chi}(h)\,dh$ is $\neq 0$.

on $\pi \subset L^2_d\left(G(F)\backslash G(\mathbb{A}_F)\right)$.

Hypothesis: Suppose from now on that

- $\varepsilon(M, \chi) = -1 = \varepsilon(M, E)$.
- $\forall v | \infty$, ϕ_v character

$\rightsquigarrow \phi_v \simeq \hat{\bigoplus_{i=1}} \mathrm{Ind}\left(\frac{z}{\bar{z}}\right)^{a_{i,v}}$

$a_{i,v} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$

$a_{1,v} > a_{2,v} > \cdots$

$\rightsquigarrow Z_v = \bigoplus_{i=1}^{m} (\mathbb{Z}/2\mathbb{Z}) \cdot \varepsilon_{i,v}$

$\rightsquigarrow c_v(\varepsilon_{i,v}) = -1$ $\quad \tau: E \to \mathbb{C}$

- Fix $v_0 | \infty$ and change c_{v_0} to $\quad (j = 1, \ldots, m)$

$$c^j_{v_0}(\varepsilon_{i,v_0}) = \begin{cases} -1 & i \neq j \\ 1 & i = j \end{cases}$$

$\rightsquigarrow$ Aut. rep. $\pi^j = \pi_f \otimes \pi^j_\infty$ of $G = SO(V, \varphi)$ of

$$\mathrm{sign}(V, \varphi) = \begin{cases} (2n-1, 2) & v = v_0 \\ (2n+1, 0) & v \neq v_0 \end{cases}$$

$\rightsquigarrow \exists\ H_0 = U(W, \psi) \subset G_0 = SO(V, \varphi)$.

$H = \mathrm{Res}_{F/\mathbb{Q}} H_0 \quad \subset \quad G = \mathrm{Res}_{F/\mathbb{Q}} G_0$.

herm. Sym domains $\quad Y \quad \subset \quad X$.

Shim var. $\quad Sh(H, Y) \quad \subset \quad Sh(G, X)$

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Reflex fields $\quad \tau_0 E \quad \supset \quad \tau_0 F$

dimensions $\quad n-1 \xrightarrow{n} 2n-1$

- $a_{i,v} = n-i+1/2 \quad \forall i,v$ (assumption on Hodge-Tate weights)

Fix $K \subset G(\mathbb{A}_F)$ compact open, $\mathcal{H}_K$ = Hecke algebra.
The $\mathcal{H}_K \otimes \mathbb{C}$ up to π_f^K is defined over $\mathcal{H}_K \otimes C$.

- $H^*(Sh_K)[\pi_f] = \mathrm{Hom}_{\mathcal{H}_K \otimes C}(\pi_f^K, H^*(Sh_K, C))$

<u>Fact:</u> $H^*(Sh_K)[\pi_f] = H^{2n-1}(Sh_K)[\pi_f] \simeq M(-n)$
$\rightsquigarrow$ M occurs in $H^{2n-1}(Sh_K)(n)$.

Fix L/F finite ext. (think $L = E[c]$). Fix λ a place of C.

$$\begin{array}{lcl}
\text{cyc}: CH^n(Sh_K \times L) & \longrightarrow & H^0(L, H^{2n}_{\acute{e}t}(\overline{Sh}_K, C_\lambda(n))) \\
\quad\quad \cup & & \\
\text{Abel Jacobi}: CH^n_0(Sh_K \times L) & \longrightarrow & H^1_f(L, H^{2n-1}_{\acute{e}t}(\overline{Sh}_K, C_\lambda(n))) \\
& & \quad\quad \downarrow \\
& & H^1_f(L, M_\lambda).
\end{array}$$

For $g \in G(\mathbb{A}_F)$, define

$$Z_K(g) = \text{image of } gK \times Y \text{ in } Sh_K(G,X)(\mathbb{C})$$
$$\| $$
$$G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K \times X$$

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$\rightsquigarrow$ This defines a set Z_K of codim n cycles.

$$H(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K \xrightarrow{\sim} Z_K$$

by strong approx:

$$\Gamma \circlearrowleft \quad {}_{H'(\mathbb{A}_f) H(\mathbb{Q})} \backslash G(\mathbb{A}_f) / K \xrightarrow{\sim} Z_K$$

Def: Let $Z_K(C) = \{ Z \in Z_K : \text{defined over } E[c] \}$.

Def: $\mathcal{N} = \{ l_1 \cdots l_r : l_i \neq l_j \text{ inert in } E/F,\ l_i \notin S \}$

S = finite set of bad places

Thm: Given $\xi(1) \in Z_K(1)$, $\exists$ $(\xi(c))_{c \in \mathcal{N}}$,

$\xi(c) \in$ Span of $Z_K(c)$ in $CH^n(Sh_K \times E[c])$ s.t.

$\forall$ $cl \in \mathcal{N}$,

$$Tr_{E[cl]/E[c]} \xi(cl) = T_{l,n} \cdot \xi(c).$$

$$\mathcal{H}\left(SO(V_l, q_l) // K_l \right) \simeq \mathbb{Z}[T_{l,1}, \dots, T_{l,n}].$$

Remark: if $T_{l,i}$ acts by $t_{l,i} \in C$ on $\pi_l^{K_l}$,

$$P_l(x) = \det_{C_\lambda}(Fr_l - X\, Id \mid M_\lambda) \in C[x]$$

$$\equiv \sum_{i=1}^{n} (-1)^i X^i (X^i - 1)^{n-i}\, t_{l,i} \pmod{N(l)+1}.$$

$\Rightarrow$ $P_l(x)$ is divisible by $X^i - 1$ mod $(N(l)+1, t_{l,n})$.