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Automorphic descent from GL(n) to classical groups:

(an outgrowth from Piatetski-Shapiro's vision)

I. Rankin-Selberg integrals:

1. $GL_n \times GL_m$ $(m<n)$: These were written down explicitly in Cogdell's talk.

2. $GL_n \times GL_n$:

π, τ irred. auto unitary cusp. reps. of $GL_n(\mathbb{A})$.
($\mathbb{A} = \mathbb{A}_F$, F = # field)

$$\int_{C_{\mathbb{A}} GL_n(F) \backslash GL_n(\mathbb{A})} \varphi_\pi(g)\varphi_\tau(g) E(f_{\omega_\pi \omega_\tau}, g)\, dg$$

$\uparrow$ Eisen. series

$$\mathrm{Ind}_{P_{n-1,1}(\mathbb{A})}^{GL_n(\mathbb{A})} \delta^{s-1/2} \chi^{-1}_{\omega_\pi \omega_\tau}$$

$$\delta\left(\begin{array}{c|c} \overset{n-1}{a} & x \\ \hline 0 & b \end{array}\right) = \frac{|\det a|}{|b|^{n-1}}$$

$$\chi_{\omega_\pi \omega_\tau}\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \omega_\pi \omega_\tau(b)$$

$$f^{\phi}_{\omega_\pi \omega_\tau, s}(g) = |\det(g)|^s \int_{\mathbb{A}^\times} \phi(t(0,\dots,1)g)|t|^{ns} \omega_\pi \omega_\tau(t)\, d^\times t$$

$$\phi \in \mathcal{S}(\mathbb{A}^n)$$

$E(f^{\phi}_{\omega_\pi \omega_\tau, s}, \cdot)$ has poles at $\mathrm{Re}(s) > 1/2 \iff \omega_\pi \omega_\tau = |\cdot|^{it}$, $t \in \mathbb{R}$

only pole $s = 1 - \frac{it}{n}$.

Normalize so $t=0$. $\Rightarrow$ $\omega_\pi\omega_\tau = 2$ to have pole.

pole at $s=1$

$$\Leftrightarrow \int_{C_{/A} GL_n(F)\backslash GL_n(A)} \varphi_\pi(g)\varphi_\tau(g)\, dg \neq 0$$

$$\Leftrightarrow \quad \tau = \overline{\pi} = \hat{\pi}.$$

$$\int_{N_n(A)\backslash GL_n(A)} W^{\psi}_{\varphi_\pi}(g) W^{\psi^{-1}}_{\varphi_\tau}(g)\, \phi((0,\ldots,0,1)g)\, |\det g|^s\, dg$$ represents $L(\pi\times\tau, s)$.

3. $\Lambda^2(GL_{2n})$: (Jacquet - Shalika)

$$\int_{C_{/A}GL_n(F)\backslash GL_n(A)} \left[\int_{M_n(F)\backslash M_n(A)} \varphi_\pi\left(\begin{pmatrix} I_n & X \\ 0 & I_n\end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g\end{pmatrix}\right)\psi(\mathrm{tr}\, X)\, dx\right] E(f^{\phi}_{\omega_\pi, s}, g)\, dg$$

gives $L^{S'}(\pi, \Lambda^2, s)$

pole at $\mathrm{Re}(s) > 1/2$ we must have $\omega_\pi = |\cdot|^{it}$, $t\in\mathbb{R}$.
Normalize to $t=0$. Then the pole is at $s=1$. We must have

Shalika period

$$\int_{GL_n(F)\backslash GL_n(A)} \int_{M_n(F)\backslash M_n(A)} \varphi_\pi\left(\begin{pmatrix} I_n & x \\ & I_n\end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g\end{pmatrix}\right)\psi(\mathrm{tr}\, x)\, dx\, dg \neq 0$$

4: $SO_{2n+1}\times GL_m$:

π = irred. auto. cusp rep. of $SO_{2n+1}(A)$

τ = irred. auto. cusp. rep. of $GL_m(\mathbb{A})$.

$m=n$:

$$SO_{2n} \hookrightarrow SO_{2n+1}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}$$

Consider

$$\mathrm{Ind}_{P_{\mathbb{A}}}^{SO_{2n}(\mathbb{A})} \tau |\det|^{s-1/2}.$$

$E(f_{\tau,s}, g)$: integrate against this

$$\int_{SO_{2n}(F)\backslash SO_{2n}(\mathbb{A})} \varphi_\pi(h)\, E(f_{\tau,s}, h)\, dh \equiv 0 \quad \text{unless } \pi \text{ is generic.}$$

If π is generic it represents $\dfrac{L^S(\pi\times\tau, s)}{L^S(\tau, \wedge^2, 2s)}$.

Unipotent radicals inside SO_ℓ:

$$U_k^\ell = \left\{ u = \begin{pmatrix} Z & X & Y \\ & I_{\ell-2k} & X' \\ & & Z^* \end{pmatrix} \in SO_\ell : Z \in N_k \right\}$$

Character $\psi_{U_k^\ell}$ of $U_k^\ell(\mathbb{A})$

$$u \longmapsto \psi_{N_k}(Z) \cdot \begin{cases} \psi(X_{k,m-k} - \frac{1}{2} X_{k,m-k+1}) & \ell = 2m \quad (k<m) \\ \psi(X_{k,m-k+1}) & \ell = 2m+1. \end{cases}$$

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The stabilizer of $\psi_{U_k^\ell}$

$$\begin{pmatrix} I_k & & \\ & h & \\ & & I_k \end{pmatrix} \qquad h\begin{pmatrix} 0 \\ \vdots \\ 1 \\ -\tfrac{1}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \ell = 2m \qquad h\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

preserves this

<u>If $m<n$:</u>

$$\int_{SO_{2m}(F)\backslash SO_{2m}(\mathbb{A})} \varphi_\pi^{\psi_{U^{2n+1}_{n-m}}}(g)\, E(f_{\tau,s}, g)\, dg$$

<u>If $m>n$:</u>

$$\int_{SO_{2n+1}(F)\backslash SO_{2n+1}(\mathbb{A})} \varphi_\pi(g)\, E^{\psi_{U^{2m}_{m-n-1}}}(f_{\tau,s}, g)\, dg$$

When they are non-vanishing, get the same ratio of L-fctns in each of these cases.

Once again, only pole for ~~[illegible]~~ is at $s=1$.

<u>II. A Case of functoriality:</u>

Assume that π lifts almost everywhere to τ - cuspidal on $GL_{2n}(\mathbb{A})$.

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$\tau = \hat{\tau}$

$\omega_\tau = 1$

$L^S(\pi \times \pi, s) = L^S(\tau \times \tau, s)$ has a pole at $s=1$

$\Rightarrow L^S(\tau, \Lambda^2, s)$ has a pole at $s=1$.

$$\langle \varphi_\pi, \operatorname{Res}_{s=1} E(f_{\tau,s}, \cdot)^{\psi_{U_{n+1}^{4n}}} \rangle_{SO_{2n+1}} \not\equiv 0.$$

$\Pi_\psi(\tau) =$ space spanned by
$$\{ g \longmapsto (\operatorname{Res}_{s=1} E(f_{\tau,s}, \cdot))^{\psi_{U_{n+1}^{4n}}} \}$$

<u>Theorem (Ginzburg-Rallis,-S.):</u> Start w/ τ irred. cusp. unitary rep. of $GL_{2n}(\mathbb{A})$ s.t. $L^S(\tau, \Lambda^2, s)$ has a pole at $s=1$. Construct $\pi_\psi(\tau)$ - a rep. of $SO_{2n+1}(\mathbb{A})$.

1) $\pi_\psi(\tau)$ is nontrivial, cuspidal, multiplicity free, and all its irred. subrep's are generic, and lift weakly to τ.
2) The 1st result works also for isobaric sums $\tau_1 \boxplus \cdots \boxplus \tau_r$, $\tau_i \neq \tau_j$, ~~cusp~~ cuspidal s.t. $L^S(\tau_i, \Lambda^2, s)$ have poles at $s=1$.
3) (w/ Jiang) $\pi_\psi(\tau)$ are irred. and the lift $\pi \to \tau$ is strong.