Langlands Functoriality and Harder's Conjecture

Dummigan

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$$G = SO(n+1, n) = \{ g \in \mathrm{Mat}_{2n+1} \mid {}^t g J g = J,\ \det g = 1 \}$$

$$J = \left[\begin{array}{c|c|c} & & I_n \\ \hline & 2 & \\ \hline I_n & & \end{array}\right]$$

Maximal torus T : typical element $\mathrm{diag}(t_1, t_2, \ldots, t_n, 1, t_1^{-1}, \ldots, t_n^{-1})$.

We have chars. $e_i : (t_1, t_2, \ldots, t_n, 1, t_1^{-1}, \ldots, t_n^{-1}) \mapsto t_i$.

These generate the character group $X^*(T)$.

Cocharacters are given by $t \xmapsto{f_1} \mathrm{diag}(t, 1, \ldots, 1, 1, t^{-1}, 1, \ldots, 1)$.

These are in the cocharacter group $X_*(T)$.

$$H = Sp_n = \{ h \in \mathrm{Mat}_{2n} \mid {}^t h J' h = J' \} \qquad J' = \begin{bmatrix} & -I_n \\ I_n & \end{bmatrix}.$$

T' = maximal torus, typical element $\mathrm{diag}(t_1, \ldots, t_n, t_1', \ldots, t_n')$.

Characters $e_i' \in X^*(T')$ given by $e_i' : \mathrm{diag}(t_1, \ldots, t_n, t_1', \ldots, t_n') \mapsto t_i$.

We have

$$H = \hat{G} = {}^L G^\circ$$

$$G = \hat{H}.$$

$A/\mathbb{Q}$ n-dim abelian variety

p prime of good reduction

$l \neq p$.

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_{A,l}} GSp_n(\mathbb{Q}_l)$$

$$\rho_{A,l}(\mathrm{Frob}_p^{-1}) \sim p^{1/2} \underbrace{\mathrm{diag}(\alpha_1, \ldots, \alpha_n, \alpha_1^{-1}, \ldots, \alpha_n^{-1})}_{s(p) \in Sp_n = \hat{G}}, \quad |\alpha_i| = 1.$$

$$L_p(A,s) = \det(I - \rho_{A,l}(\mathrm{Frob}_p^{-1})\, p^{-s})^{-1}$$

$$L_p(A, s + 1/2) = \det(I - s(p)\, p^{-s})^{-1}.$$

Consider

$$\sigma_p : T(\mathbb{Q}_p) \to \mathbb{C}^\times,$$

with $\sigma_p(T(\mathbb{Z}_p)) = \{1\}$.

Then

$$\sigma_p(f_i(p)) = \alpha_i$$

for some α_i in the unit circle. Can define σ_p to recover the α_i above. Extend this to the Borel subgroup $B(\mathbb{Q}_p)$ (trivial on $U(\mathbb{Q}_p)$); then induce to $G(\mathbb{Q}_p)$ to get a unitary rep. Π_p.

Conjecturally, $\otimes \Pi_p$ is part of a cuspidal auto. rep. of $G(\mathbb{A})$.

Special case:

$n=1 \qquad SO(2,1) \simeq PGL_2$ [modularity of elliptic curves]

$n=2 \qquad SO(3,2) \simeq PGSp_4$.

These two are "accidental" isomorphisms, in general one uses $SO(n+1,n)$.

Choose a maximal compact in G, P, by requiring it to have Levi subgroup $M \simeq GL_1 \times SO(n,n-1)$. Denote the unipotent radical by N.

In $\hat{G} = Sp_n$, the corresponding maximal parabolic $\hat{P}$ is given with Levi $\hat{M} \simeq GL_1 \times Sp_{n-1}$. Denote the unipotent radical by $\hat{N}$. For $\hat{P}$, we have

$$(t, [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]) \longmapsto \left[\begin{array}{c|c|c|c} t & & & \\ \hline & A & & B \\ \hline & & t^{-1} & \\ \hline & C & & D \end{array}\right]$$

If we include $\hat{N}$:

$$\left[\begin{array}{c|c|c|c} t & {}^t v_2 & * & {}^t v_1 \\ \hline 0 & A & v_1 & B \\ \hline 0 & 0 & t^{-1} & 0 \\ \hline 0 & C & v_2 & D \end{array}\right]$$

Let Π' be a cusp. auto rep. of $SO(n,n-1)$. Dummigan

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$\Pi = 1 \times \Pi'$ of $M \simeq GL_1 \times SO(n,n-1)$. At an unramified prime

p, $\Pi_p \rightsquigarrow (1, s(p)) \in \hat{M}$.

Adjoint $r : \hat{M} \to Aut(\hat{N})$ (act via conjugation)

$r = r_1 \oplus r_2$

$st \oplus 1$.

from action on $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ from what happens to $*$.

$$L_p(s, \Pi_p, r_1) = \prod_{i=1}^{n-1} (1 - \alpha_i p^{-s})^{-1} (1 - \alpha_i^{-1} p^{-s})^{-1}$$ (standard L-fctn for orthogonal group. (Spin for $n=2$ if consider Sp_2))

$$L_p(s, \Pi_p, r_2) = (1 - p^{-s})^{-1}$$ ($\zeta(s)$.)

Conjecture: Suppose $\mathrm{ord}_q(L_{alg}(1+s, \Pi, r_1)) > 0$

(or same statement w/ r_2), then $\exists$ tempered cusp. auto. rep.

$\tilde{\Pi}$ of $G(\mathbb{A})$ s.t. $\tilde{S}(p) \equiv \underbrace{(p^{-s}, s(p))}_{\text{Satake parameter of an induced rep.}} \pmod{q}$.

If Π_∞ has infinitesimal character $a_1 e_2 + \cdots + a_{n-1} e_n$, then

$\tilde{\Pi}_\infty$ has an infinitesimal character $a_1 e_1 + a_2 e_2 + \cdots + a_{n-1} e_{n-1} + s e_n$,

$a_1 > a_2 > \cdots > a_{n-1} > s > 0$, $a_i, s \in 1/2 + \mathbb{Z}$, (excl. $s = 1/2$).

($n=2$, $s=1/2$ is CAP. Otherwise the induction are not cuspidal!)

Example: $G = SO(3,2) \simeq PGSp_4$

$M \simeq GL_1 \times SO(2,1) \simeq GL_1 \times PGL_2$.

f cuspidal Hecke eigenform of wt k'. Π_∞ inf. char. $\frac{k'-1}{2} e_2$

$0 < s < \frac{k'-1}{2}$. Say $s = \frac{j+1}{2}$, j even, $j > 0$.

$k' = j + 2k - 2$, $k \geq 3$.

$s(p) \sim \mathrm{diag}(\alpha(p), \alpha(p)^{-1})$.

$$L(1+s, \Pi, r_1) = L_f(1 + s + \tfrac{k'-1}{2})$$

$$= L_f(j+k).$$

$\tilde{\Pi}_\infty = \frac{j+2k-3}{2} e_1 + \frac{j+1}{2} e_2 \rightsquigarrow$ vector valued Siegel modular form of degree 2 of wt $\mathrm{Sym}^j \det^k$.

$$T(p) \text{ acts by } p^{\frac{j+2k-3}{2}} \mathrm{tr}(\tilde{S}(p)) \equiv p^{\frac{j+2k-3}{2}} (p^{-s} + \alpha(p) + p^{s} + \alpha(p)^{-1})$$

$$\equiv a_p(f) + p^{k-2} + p^{j+k-1}. \pmod{\eta}.$$

For $j > 0$ This is Harder's conjecture in this setting.

For $n=1$, we recover the congruences given by primes dividing Bernoulli numbers.