Local and Global Maass Relations

(joint w/ Saha and Schmidt)

Saito-Kurokawa lifts:

$$f \in S_{2k-2}(SL_2(\mathbb{Z})) \longrightarrow F \in S_k(Sp_4(\mathbb{Z})).$$

k even

$$F = \sum_{S \in P_2(\mathbb{Z})} A(S)\, e^{2\pi i \, Tr(SZ)}$$

Maass relations

$$A\left(\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}\right) = \sum_{d \mid (a,b,c)} d^{k-1} A\left(\begin{bmatrix} \frac{ac}{d^2} & b/2d \\ b/2d & 1 \end{bmatrix}\right).$$

$$\begin{tikzcd} \pi_f & PGL_2(\mathbb{A}) & \widetilde{SL}_2(\mathbb{A}) \arrow[l, "\text{Wald}"'] \arrow[r, "\text{Theta}"] & PGSp_4(\mathbb{A}) \arrow[d] \\ f \arrow[u] & & & F \end{tikzcd}$$

If you forget the forms f and F, you can still do the lift but you lose info. on f.c. This is important because it is believed f.c. contains more info than e.v.

SK lift: Let $f \in S_{2k-2}(SL_2(\mathbb{Z}))$ with k even, $\exists$ $F \in S_k(Sp_4(\mathbb{Z}))$ s.t.

$$L(s, F) = L(s, f)\, \zeta(s + 1/2)\, \zeta(s - 1/2).$$

Idea:

$$\begin{array}{ccc} F \mapsto \Phi_F \in & \pi_F = \otimes' \pi_p \\ \downarrow & \wr \\ \int^S \Phi_F & \text{Bessel models} \\ \downarrow & \prod_{p<\infty} B_p \\ A(s) & \longleftrightarrow \end{array}$$

Bessel models:

F local field , $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$

$$\vartheta_S : \overset{\in U(F)}{\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}} \longrightarrow \psi(\mathrm{tr}(SX))$$

$X \in \mathrm{Sym}_2$

$$T_S = \{ g \in GL_2(F) : {}^t g S g = \det(g) S \}$$

$$\hookrightarrow \begin{bmatrix} g & \\ & \det(g)\, {}^t g^{-1} \end{bmatrix}$$

If disc(S) is a square, $L = F \oplus F$ (split case);

if it is not a square $L = F(\sqrt{\mathrm{disc}(S)})$.

Bessel subgroup $R = T_S U$

$\Lambda \otimes \vartheta_S$: $tu \longmapsto \Lambda(t)\vartheta_S(u)$

where Λ is any char. of $L^\times$.

Let π be an irred. admiss. rep. of $GSp_4(F)$.

Bessel functional: $\beta : V_\pi \longrightarrow \mathbb{C}$

$$\beta(\pi(r)v) = \Lambda\otimes\vartheta_S(r)\,\beta(v).$$

$v \in V_\pi$

$$B_v : G(F) \to \mathbb{C}$$
$$B_v(g) = \beta(\pi(g)v).$$

These satisfy $B_v(\cancel{g}\,rg) = \Lambda\otimes\vartheta_S(r)\,B_v(g)$. $(*)$.

Given Bessel model for π: $B \in V_\pi \longmapsto B(1)$.

(The functional and model go hand in hand.)

Consider π to be spherical, i.e., $V_\pi^{GSp_4(\mathcal{O}_F)} \neq 0$.

$B \in V_\pi^{GSp_4(\mathcal{O}_F)}$ is determined by its value on $R(F)\backslash GSp_4(F)/GSp_4(\mathcal{O}_F)$.

<u>Assumption</u>: $a, b \in \mathcal{O}_F$, $c \in \mathcal{O}_F^\times$ $\left(\text{w/ } S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \right)$

$(*)$
- if L/F field, disc(F) is generator of disc(L/F)
- if $L = F\oplus F$, disc$(F) \in \mathcal{O}_F^\times$.

If these are satisfied, then

$$GL_2(F) = \bigsqcup_{m\geq 0} T_S(F)\begin{bmatrix} \varpi^m & 0 \\ 0 & 1 \end{bmatrix} GL_2(\mathcal{O}_F).$$

Then

$$GSp_4(F) = \coprod_{\substack{l,m \in \mathbb{Z} \\ m \geq 0}} R(F)\, h(l,m)\, K$$

where

$$h(l,m) = \begin{bmatrix} \varpi^{l+2m} & & & \\ & \varpi^{l+m} & & \\ & & 1 & \\ & & & \varpi^{m} \end{bmatrix}$$

Sugano:

$$\sum_{l,m \geq 0} B(h(l,m))\, x^m y^l = \frac{H(x,y)}{P(x)\, Q(y)}.$$

$\Rightarrow B(1) \neq 0$ if Λ unramified.

Normalize $B(1) = 1$.

Local Maass Relation:

π is spherical rep with trivial central char. Assume π has a $1 \otimes \vartheta_S$ Bessel model for some S. Let $B \in \mathcal{B}_\pi^K$ s.t. $B(1) = 1$. T.F.A.E. :

1) One of the Satake parameters of π is $q^{\pm 1/2}$

2) $\forall\, l, m \geq 0$

$$B(h(l,m)) = \sum_{i=0}^{l} q^{-i}\, B(h(0, l+m-i))$$

(local Maass relations)

(other equivalent conditions are given in the paper.)

Global:

$$F \in S_k(Sp_4(\mathbb{Z})) \longmapsto \Phi_F$$

$$\Phi_F^S(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \Phi_F(ug)\,\vartheta_S^{-1}(u)\,du$$

$$\Rightarrow \Phi_F^S(1) = e^{-2\pi \operatorname{Tr}(S)} A(S).$$

$$\int_{T_S(\mathbb{Q})\backslash T_S(\mathbb{A})} \Phi_F^S(tg)\,\Lambda^{-1}(t)\,dt \qquad \text{Global Bessel model}$$

This can be related to local Bessel models. This is not useful though b/c it is a sum of f.c. so can't pick out just one f.c. However, the special nature of SK lifts allows us to get information from this we could not get for general Siegel modular forms.

Let F be a SK lift, Hecke e.f. This generates

$$\pi_F = \bigotimes_{p \le \infty} \pi_p .$$

π_∞ : holo. discrete series (k)

$p < \infty$ π_p is of type IIb.

<u>Fact:</u> On V_{π_p} consider fctns $\beta_p : V_p \to \mathbb{C}$ s.t.
$(p < \infty)$

$$\beta_p(\pi(u)v) = \vartheta_S(u)\, \beta_p(v) \quad \forall u \in U(\mathbb{Q}_p).$$

We have

1) The space of such functionals is 1-dim.

2) This automatically satisfies

$$\beta(\pi(m)v) = \beta(v) \quad \forall m \in T_S(\mathbb{Q}_p).$$

<u>Cor:</u> $\Phi_F^S(mg) = \Phi_F^S(g) \quad \forall\ m \in T_S(\mathbb{A})$.

$$\Phi_F^S(g) = C_S \prod_{p \le \infty} B_p(g_p)$$

<u>Normalizations:</u>

$p < \infty$, S satisfies $(*)$ then $B_p^S(1) = 1$.

if S does not satisfy $(*)$, let $S' = v\, {}^tASA$ w/
$v \in \mathbb{Q}^\times$, $A \in GL_2(\mathbb{Q})$ satisfy $(*)$. $B_p^S(1) = B_p^{S'}\left(\begin{bmatrix} A & \\ & v\,{}^tA^{-1} \end{bmatrix}\right)$

$$B_\infty^S(1) = B_\infty^{1_2}(*) = (\det S)^{k/2} e^{-2\pi \operatorname{Tr}(S)}.$$

So C_s is well-defined. It is not easy to see what it is. For instance, it contains the info of the half-integral weight form. However, it doesn't matter for the Maass relations.

$$S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \qquad \gcd(a,b,c) = 1 \qquad \mathrm{disc}(S) = N^2 L$$

$$S^{(d)} = \begin{bmatrix} ac/d^2 & b/2d \\ b/2d & 1 \end{bmatrix}.$$

$$\cancel{e^{-2\pi \mathrm{Tr}(S)}} A(S) = C_S \prod_{p \le \infty} B_p^S(1)$$

$$= C_S \det(S)^{k/2} \cancel{e^{-2\pi \mathrm{Tr}(S)}} \prod_{p \mid L} B_p^{S'}(h(v_p(L), v_p(N/L)))$$

$$\cdot \prod_{\substack{p \nmid L \\ p \mid N}} B_p^{S'}(h(0, v_p(N)))$$

$$\updownarrow$$

$$A(S^{(d)}) = C_{S^{(d)}} \det(S^{(d)})^{k/2} \prod(-) \prod(-)$$

Now solve each for C_S and $C_{S^{(d)}}$ and equate them ($C_S = C_{S^{(d)}}$!)

This gives

$$\sum_{d \mid L} d^{k-1} A(S^{(d)}) \prod_{p \mid L} B_p^{S'}(h(v_p(L), v_p(N/L)))$$

$$= \sum_{d \mid L} \frac{A(S)}{d} \prod_{p \mid L} B_p^{S'}(h'(0^*, v_p(N))).$$

Now just show

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$$\prod_{p|L} B_p^{s'}\left(h(v_p(L), v_p(N/L))\right)$$

$$= \frac{1}{d} \prod_{p|L} B_p^{s'}\left(h(0, v_p(N))\right).$$

This is true by the local Maass relations.