

Siegel modular forms of degree 2: Fourier coefficients, L-functions, and functoriality (a survey)

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14th July 2014

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 - Hecke operators and L -functions
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- 4 Bocherer's conjecture and refinements
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Definition of Sp_4

For a commutative ring R , we denote by $\mathrm{Sp}_4(R)$ the set of 4×4 matrices $A \in \mathrm{GL}_4(R)$ satisfying the equation $A^t J A = J$ where

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Definition of \mathbb{H}_2

Let \mathbb{H}_2 denote the set of complex 2×2 matrices Z such that $Z = Z^t$ and $\mathrm{Im}(Z)$ is positive definite.

\mathbb{H}_2 is a homogeneous space for $\mathrm{Sp}_4(\mathbb{R})$ under the action

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto (AZ + B)(CZ + D)^{-1}$$

Siegel modular forms

A Siegel modular form of degree 2, full level and weight k is a holomorphic function F on \mathbb{H}_2 satisfying

$$F(\gamma Z) = \det(CZ + D)^k F(Z),$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z})$,

If in addition, F vanishes at the cusps, then F is called a **cuspidal form**.

We define $S_k(\mathrm{Sp}_4(\mathbb{Z}))$ to be the space of cuspidal forms as above.

Remark. The smallest k for which $S_k(\mathrm{Sp}_4(\mathbb{Z}))$ is non-zero is $k = 10$.

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The space $S_\rho(\Gamma)$

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- 1 $F(\gamma Z) = \rho(CZ + D)F(Z)$, for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$,
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Two remarks:

- $\rho \simeq \det^l \mathrm{sym}^m$ for some integers l, m .
- If $\rho = \det^k$, then $V = \mathbb{C}$, and we get the usual space $S_k(\Gamma)$ of scalar valued weight k cusp forms.

$$\mathcal{P}_2 = \left\{ S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} : a, b, c \in \mathbb{Q}, S > 0 \right\},$$

$$\mathcal{P}_2(\mathbb{Z}) := \left\{ S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} : a, b, c \in \mathbb{Z}, S > 0 \right\},$$

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The Fourier expansion

Let $F(Z) \in S_\rho(\Gamma)$. Then we can write

$$F(Z) = \sum_{S \in \mathcal{P}_2} a(F, S) e^{2\pi i \operatorname{Tr} SZ}, \quad a(F, S) \in V.$$

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Two remarks:

- There exists N such that $a(F, S) = 0$ unless $S \in (1/N)\mathcal{P}_2(\mathbb{Z})$.
- There exists a congruence subgroup $\Gamma' \in \operatorname{SL}_2(\mathbb{Z})$ such that $a(F, A^t S A) = a(F, S)$ for all $A \in \Gamma, S \in \mathcal{P}_2$.

(If $\Gamma = \operatorname{Sp}_4(\mathbb{Z})$, then $N = 1, \Gamma' = \operatorname{SL}_2(\mathbb{Z})$)

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- There is a Hecke-invariant subspace of $S_k(\text{Sp}_4(\mathbb{Z}))$. (spanned by eigenforms called Saito-Kurokawa lifts.)
- *Most forms are non-lifts.* For example the Saito-Kurokawa space has dimension $\asymp k$ while $\dim(S_k(\text{Sp}_4(\mathbb{Z}))) \asymp k^3$.

Let \mathbb{A} be the adèles of \mathbb{Q} and $\pi = \otimes_v \pi_v$ be a cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$.

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$$\pi_p \simeq \pi(\chi_0, \chi_1, \chi_2)$$

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- Two possibilities for π_∞ of interest to us are $\pi_\infty \simeq L(k, l)$ (holomorphic discrete series) and $\pi_\infty \simeq L(k, -l)$ (large discrete series) where $k \geq l \geq 0$ are integers. *These are in the same L -packet.*

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Remarks:

- Multiplicity one for π is expected to be true. (Not known at present, but may follow from Arthur)
- Strong multiplicity one for π is FALSE.

Models

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- An alternative to Whittaker models is provided by the **Bessel model**. These are parametrized by characters Λ of quadratic extensions K .
- If π has a particular Bessel model then so does each π_v . But all π_v having Bessel models DOES NOT imply that π does.
- The Gan-Gross-Prasad conjectures predict that if π is tempered, and each π_v has a Bessel model, then π has a Bessel model if and only if $L(1/2, \pi_K \times \Lambda) \neq 0$.

L -functions

The Langlands L -functions

Given a cuspidal automorphic representation π of $G(\mathbb{A})$ and a **finite dimensional representation** r of the **dual group** ${}^L G^0$, there exists a global **Langlands L -function** $L(s, \pi, r)$. It is **Eulerian**, has degree r at almost all places, and (conjecturally) has a **functional equation** $s \mapsto 1 - s$, (conjecturally) **no poles** except in anomalous cases.

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If $G = \mathrm{GSp}_4$, then ${}^L G^0 \simeq \mathrm{GSp}_4(\mathbb{C})$. The two smallest dimensional non-trivial irreducible r we can get are of dimensions 4 and 5.

- $r = \rho_4$. In this case $L(s, \pi, \rho_4)$ is called the spinor L -function.
- $r = \rho_5$. In this case $L(s, \pi, \rho_5)$ is called the standard L -function.

Remark: The analytic properties of $L(s, \pi, \rho_4)$ and $L(s, \pi, \rho_5)$ are essentially known.

- Let $F \in S_\rho(\Gamma)$
- Can lift to a V -valued function $\widetilde{\Phi}_F$ on $\mathrm{Sp}_4(\mathbb{R})$ via

$$\widetilde{\Phi}_F(g) = \rho^{-1}(J(g, il_2))F(g(il_2)).$$

- Now we extend $\widetilde{\Phi}_F$ to a V -valued function on $\mathrm{GSp}_4(\mathbb{A})$ via **strong approximation**.

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For the last step, we pick local subgroups K'_p of $\mathrm{GSp}_4(\mathbb{Z}_p)$ such that

- 1 $\mu_2 : K'_p \mapsto \mathbb{Z}_p^\times$ is surjective,
- 2 $\mathrm{GSp}_4(\mathbb{R}) \prod_p K'_p \cap \mathrm{GSp}_4(\mathbb{Q})^+ \subset \Gamma$.

Then $\widetilde{\Phi}_F$ is right invariant under each K'_p .

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- $\pi_F = \bigoplus_{i=1}^t \pi_F^{(i)}$ with each $\pi_F^{(i)}$ an irreducible, cuspidal automorphic representation.

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- If F is an eigenfunction of the Hecke algebra at every place, then $t = 1$.

Summary: Adelization and deadelization

- Any vector-valued Siegel cusp form F leads to an adelic function Φ_F and then to a (not-necessarily irreducible) cuspidal automorphic representation π_F .
- If F is an eigenfunction of all local Hecke algebras then π_F is irreducible (but π_F may be irreducible even without this).
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- π_F is NOT generic (fails at infinity).
- **De-adelization:** Suppose π is an irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, with $\pi_\infty \simeq L(k, l)$. Then each vector $\Phi \in \pi$ of suitable type, gives rise to a $F \in S_{\det' \mathrm{sym}^{k-l}}(\Gamma)$ for some suitable Γ .
- If π_p is generic at all finite places, can pick F uniquely up to multiples so that Γ is **paramodular subgroup** of correct level.

One can attach Galois representations to Siegel cusp forms.

Theorem (Weissauer)

Let π be an irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ such that $\pi_\infty \simeq L(k, l)$. Then there exists a Galois representation

$$\rho_{\pi, \lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_4(E_\lambda)$$

such that for almost all primes p ,

$$\mathrm{Tr}(\rho_{\pi, \lambda}(\mathrm{Fr}_p)) = a(\pi_p).$$

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Corollary (Kowalski-S, 2013)

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform at all primes. Then the set of primes where the Hecke eigenvalue $a_{F, p}$ is 0, has density 0.

Functoriality

Let G, H be connected reductive groups, $u : {}^L H^0 \rightarrow {}^L G^0$ a homomorphism. Then, functoriality predicts that given any automorphic representation π on $H(\mathbb{A})$, there exists an automorphic representation π' on $G(\mathbb{A})$ such that for all finite dimensional representations r of ${}^L G^0$,

$$L(s, \pi, u \circ r) = L(s, \pi', r).$$

Theta lifts

Given a symplectic space W and an orthogonal space V , there is a theta correspondence that takes automorphic representations of $\mathrm{Sp}(W)$ or $\widetilde{\mathrm{Sp}}(W)$ to automorphic representations on $\mathrm{SO}(V)$ (and vice-versa).

Using $\mathrm{PD}^\times \simeq \mathrm{SO}(3)$ and $\mathrm{PGSp}_4 \simeq \mathrm{SO}(5)$, we have

$$\begin{array}{ccccc}
 \pi & \mathrm{PD}^\times(\mathbb{A}) & \xleftarrow{\text{Wald}} & \widetilde{\mathrm{SL}}(\mathbb{A}) & \xrightarrow{\theta} & \mathrm{PGSp}_4(\mathbb{A}) & \Pi \\
 \uparrow & & & & & & \downarrow \\
 f & & & & & & F
 \end{array} \quad (1)$$

This allows us to take a classical cusp form f of weight $2k - 2$ for $\Gamma_0(N)$, and produce a Siegel cusp form $F \in S_k(\Gamma)$ for some Γ .

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This allows us to take a classical cusp form f of weight $2k - 2$ for $\Gamma_0(N)$, and produce a Siegel cusp form $F \in S_k(\Gamma)$ for some Γ . For this to work, we need

$$(-1)^{|S|} = \varepsilon(1/2, \pi_f),$$

where S is a set of places including ∞ where π_f is discrete series. If f has full level, we must have k even, and we recover the classical Saito-Kurokawa lift $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$.

The Saito-Kurokawa lift and functoriality

How does the correspondence $\pi_f \mapsto \Pi$ from automorphic representations of PGL_2 to PGSp_4 fit with functoriality?

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We have a map of L -groups

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Theorem (Schmidt (2005))

Π is the functorial lift of $\pi_f \otimes \pi_S$ under the above embedding, where π_S is the unique subquotient of $\mathrm{Ind}(\|^{1/2}, \|-^{-1/2})$ that is unramified exactly at the places outside S .

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Theorem (Andrianov, Piatetski-Shapiro, Weissauer, Pitale-Schmidt)

Suppose that $k > 2$ and $F \in S_k(\Gamma)$ generates an irreducible cuspidal automorphic representation Π of $\mathrm{PGSp}_4(\mathbb{A})$. Then the following are equivalent.

- ① *F is a Saito-Kurokawa lift, or a quadratic twist of it.*
- ② *Π is CAP with respect to the Siegel parabolic.*
- ③ *Π is CAP.*
- ④ *Π is non-tempered at some unramified prime (i.e., F does not satisfy the Ramanujan bound)*
- ⑤ *$L(s, \Pi \times \chi, \rho_4)$ has a pole for some quadratic character χ .*

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Under suitable conditions, the resulting lift is non-zero, cuspidal, and of the form $L(k, l)$ at infinity. Forms F obtained from such Π are called endoscopic lifts, or Yoshida lifts.

Various lifts to $S_\rho(\Gamma)$

- Saito-Kurokawa (CAP) lifts F , with strange properties. For these, there exists a classical newform f and a quadratic character χ (possibly trivial) such that

$$L(s, \Pi_F \otimes \chi) \approx L(s, \pi_f) \zeta(s + 1/2) \zeta(s - 1/2)$$

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- Lifts from $GL_2(K)$, where K is a quadratic field. These may be viewed as the non-split version of Yoshida lifts. See papers of Roberts–Johnson–Leung (real quadratic) and Berger–Dembele–Pacetti–Sengun (imaginary quadratic).

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This should follow (follows?) from the work of Arthur using methods of the trace formula. However, using the converse theorem, Pitale, Schmidt and I proved:

Theorem (Pitale–Schmidt–S, 2012)

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be an eigenform for all Hecke operators, and not of Saito-Kurokawa type. Let π_F be the associated cuspidal, automorphic representation of $GS\mathrm{p}_4(\mathbb{A})$. Then π_F admits a strong lifting to an automorphic representation Π_4 of $GL_4(\mathbb{A})$, and a strong lifting to an automorphic representation Π_5 of $GL_5(\mathbb{A})$. Both Π_4 and Π_5 are cuspidal.

Method

We prove the following theorem and then apply the converse theorem

Theorem

*Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform that is not a Saito-Kurokawa lift and π be **any** cuspidal automorphic representation of GL_2 . Then $L(s, F \times \pi)$ is absolutely convergent for $\mathrm{Re}(s) > 1$, has **meromorphic continuation** to the entire complex plane, and the completed L-function satisfies the usual **functional equation**, is **entire**, and **bounded in vertical strips**.*

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The starting point for this is an integral representation due to Furusawa. We generalize Furusawa's formula, use this generalization to prove meromorphic continuation, functional equation and boundedness, and then prove a pullback formula and a seesaw argument to prove entireness.

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Corollary

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be an eigenform for all Hecke operators, and not of Saito-Kurokawa type. Let π_F be the associated cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. Let π'_F be the representation obtained by switching the Archimedean $L(k, k)$ to $L(k, -k)$. Then π'_F is also cuspidal automorphic (and now also generic!).

Fourier coefficients are mysterious objects

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be an eigenform. Assume k even. Recall the Fourier expansion

$$F(Z) = \sum_{S \in \mathcal{P}_2(\mathbb{Z})} a(F, S) e^{2\pi i \mathrm{Tr}SZ}.$$

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If $\mathrm{disc}(S_1) = d_1 \neq d_2 = \mathrm{disc}(S_2)$ are two such fundamental discriminants, then we cannot understand one from the other via Hecke operators.

Put $K = \mathbb{Q}(\sqrt{d})$ and let Cl_K denote the ideal class group of K . Then $\text{SL}_2(\mathbb{Z})$ -equivalence classes of binary quadratic forms of discriminant d are in natural bijective correspondence with the elements of Cl_K . Define

$$R(f, K) = \sum_{c \in \text{Cl}_K} a(f, c). \quad (2)$$

Bocherer's conjecture

There exists a constant c_f depending only on f such that for any imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with $d < 0$ a fundamental discriminant, we have

$$|R(f, K)|^2 = c_f \cdot |d|^{k-1} \cdot w(K)^2 \cdot L(1/2, \pi_f \times \chi_d).$$

Thus, Bocherer's conjecture predicts that the sum of Fourier coefficients of discriminant d is essentially a L -value.

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- 1 What is *exactly* the constant c_f ?
- 2 Instead of a plain sum, what if we weigh them by a character Λ ?
- 3 What is the proper generalization to the case of $S_\rho(\Gamma)$, or to general automorphic representations π of $\mathrm{GSp}_4(\mathbb{A})$?

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All of these have now been addressed by very general and exact conjectures (Furusawa–Martin–Shalika, Prasad–Takloo-Bighash, Gan–Gross–Prasad, Liu,...).

Let π be an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. For any automorphic form ϕ in the space of π , we can define a **global Bessel period** $B(\phi, \Lambda)$ on $\mathrm{GSp}_4(\mathbb{A})$ by

$$B(\phi, \Lambda) = \int_{\mathbb{A}^\times T_S(F) \backslash T_S(\mathbb{A})} \int_{N(F) \backslash N(\mathbb{A})} \phi(tu) \Lambda^{-1}(t) \theta_S^{-1}(n) dn dt. \quad (3)$$

where S is a symmetric matrix, K a quadratic extension attached to S , Λ a Hecke character of K , $T_S \simeq K^\times$ the non-split torus of GL_2 .

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Two key points:

- π has a Bessel model of type (K, Λ) if and only if $B(\phi, \Lambda) \neq 0$ for some ϕ .
- If ϕ is the adelization of some $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$, and Λ corresponds to a character of Cl_K , then

$$B(\phi, \Lambda) = e^{-2\pi \mathrm{Tr}(S)} \sum_{c \in \mathrm{Cl}_K} \Lambda^{-1}(c) a(F, c).$$

A conjecture of Yifeng Liu

Let π, Λ be as above. Suppose that for almost all places v of F , the local representation π_v is generic. Let ϕ be any automorphic form in the space of π . Then

$$\frac{|B(\phi, \Lambda)|^2}{\langle \phi, \phi \rangle} = \frac{C_T \zeta(2)\zeta(4)L(1/2, \pi \otimes \mathcal{AI}(\Lambda))}{4 L(1, \text{sym}^2 \pi)L(1, \chi_d)} \prod_v I_v(\phi_v).$$

where $I_v(\phi_v)$ is an explicit **local integral**, equal to 1 almost everywhere.

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Remarks:

- This formulation excludes the Saito-Kurokawa (CAP) lift.
- Liu proved this conjecture for all endoscopic lifts.
- Note that the conjecture implies that

$$B(\phi, \Lambda) \neq 0 \Rightarrow L(1/2, \pi \otimes \mathcal{AI}(\Lambda)) \neq 0$$

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Compute $I_V(\phi_V)$ in some specific ramified cases, and thus formulate the precise refinement of Bocherer's conjecture for various Siegel cusp forms with level.

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Thank you!