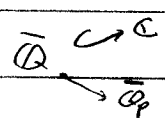
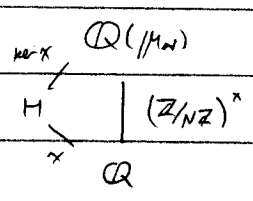


Hida families and Gross-Stark units over totally real fields:



$$\chi: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \bar{\mathbb{Q}}^\times \quad \chi(-1) = -1$$



$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1} \quad \text{for } \text{Re}(s) > 1$$

Fix a prime p s.t. $\chi(p) = 1$ (i.e., p splits completely in H)
 $\chi(a) = 0$ if $\text{gcd}(a, N) \neq 1$.

Let $\omega: G_{\mathbb{Q}} \rightarrow \mu_{p-1}$ be the Teichmüller character.

p -adic L -function $L_p(\chi\omega, s): \mathbb{Z}_p \rightarrow E = \mathbb{Q}_p(\text{values } \chi) =: \mathbb{Q}_p(\chi)$

s.t. $L_p(\chi\omega, 1-k) = L(\chi\omega^{1-k}, 1-k)$ for $k \in \mathbb{Z}, k \geq 1$.

has modulus divisible by p .
 Lies in $\mathbb{Q}(\chi, \mu_{p-1})$ e.g., $\chi\omega^0(p) = 0$, not 1.

$$L_p(\chi\omega, 0) = L(\chi\omega^0, 0)$$

$$L(\chi\omega^0, s) = (1 - \chi(p)p^{-s}) L(\chi, s)$$

$$\begin{aligned} L(\chi\omega^0, 0) &= (1 - 1 \cdot 1) \cdot L(\chi, 0) \\ &= 0. \end{aligned}$$

Def: $Z_{an}(X) = \frac{L_p(XW, 0)}{L(X, 0)} \in E$

$L(X, 0) \neq 0$ since $X(-1) = -1$

Let $U = \{u \in H^* : |u|_w = 1 \ \forall w \in P\}$. This is a finitely
(including arch. primes)

generated abelian group of rank $[H:\mathbb{Q}]/2$.

$U_X = (U \otimes E)^{X^{-1}} = \{u \in U \otimes E : \sigma(u) = u \otimes X^{-1}(\sigma) \ \forall \sigma \in G_{\mathbb{Q}}\}$.

$\dim_E U_X = 1$. Let u_X be a generator of U_X

$U \subset H^* \subset \mathbb{Q}_p^*$ (...) $(\log_p(p) = 0)$
 $\begin{matrix} \text{ord}_p \nearrow \mathbb{Z} \\ \log_p \searrow \mathbb{Z}_p \end{matrix}$

Tensor with E to obtain:

$U \otimes E \begin{matrix} \xrightarrow{\text{ord}_p} \\ \xrightarrow{\log_p} \end{matrix} E$

Def: $Z_{alg}(X) = -\log_p(u_X) / \text{ord}_p(u_X) \leftarrow \neq 0$

Thm (Gross): $Z_{an}(X) = Z_{alg}(X)$.

Gross' proof is explicit, using formula for u_X in terms of
Gauss sums.

Gross-Koblitz formula relates Gauss sums Γ_p , p -adic Γ -function.

Ferrero-Greenberg theorem relates Γ_p to L_p . This gives the proof.

We'll give a different proof using Ribet's method.

One has the same conjecture for totally real fields. However, since there is no explicit CRT here Gross' method does not generalize, which is why this new method is nice.

Ribet's method, Step 1 Reformulation:

Kummer Theory: $\bar{U} = \{u \in \bar{\mathbb{Q}}^\times : |u|_w = 1 \text{ for all } w \neq p\}$

$$1 \longrightarrow \mu_{p^n} \longrightarrow \bar{U} \xrightarrow{p^n} \bar{U} \longrightarrow 1$$

Let $\mathcal{O} = \mathbb{Z}[X]$. Tensor the short exact sequence by $\mathcal{O}(X)$:

$$1 \longrightarrow \mu_{p^n} \otimes \mathcal{O}(X) \longrightarrow \bar{U} \otimes \mathcal{O}(X) \xrightarrow{p^n} \bar{U} \otimes \mathcal{O}(X) \longrightarrow 1.$$

Take $G_{\mathbb{Q}}$ -cohomology:

$$0 \longrightarrow (U \otimes \mathcal{O}) / \mathfrak{p}^n \longrightarrow H^1(G_{\mathbb{Q}}, \mu_{p^n} \otimes \mathcal{O}(X)) \longrightarrow H^1(G_{\mathbb{Q}}, \bar{U} \otimes \mathcal{O}(X))[\mathfrak{p}^n] \longrightarrow 0.$$

Take \varprojlim_n and then $\otimes_{\mathcal{O}} E$.

$$\begin{array}{ccccccc}
 & & H_p(G_{\mathbb{Q}}, E(x)(1)) & \longrightarrow & \text{"Image"} & \longrightarrow & 0 \\
 & \nearrow & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & (U \otimes E)^{x^{-1}} & \longrightarrow & H^1(G_{\mathbb{Q}}, E(x)(1)) & \longrightarrow & T_p H^1(G_{\mathbb{Q}}, \bar{U} \otimes \mathcal{O}(x)) \otimes_{\mathcal{O}_E} E \longrightarrow 0 \\
 & & \downarrow & & \downarrow \alpha & & \\
 & & \bigoplus_{l \neq p} H^1(G_l, E(x)(1)) & \longrightarrow & \bigoplus_{l \neq p} T_p(H^1(G_l, \bar{U} \otimes \mathcal{O}(x))) \otimes_{\mathcal{O}_E} E & \longrightarrow & 0
 \end{array}$$

$$H_p(G_{\mathbb{Q}}, E(x)(1)) = \{ [k] \in H^1(G_{\mathbb{Q}}, E(x)(1)) : \text{res}_{G_l}^{\mathbb{Q}} [k] = 0 \ \forall l \neq p \}$$

$$\text{"Image"} \subseteq \ker \alpha = T_p(\ker(H^1(G_{\mathbb{Q}}) \rightarrow \bigoplus_{l \neq p} H^1(G_l))) \otimes_{\mathcal{O}_E} E = 0$$

finite (use inflation-rest. for $G_H \subset G_{\mathbb{Q}}$, use fact that for units, $\mathbb{H} = \text{class group}$)

"Neukirch-Schmidt-Winberg"
"Cohomology of # fields" Prop 8.3.10(ii)

$$\Rightarrow (U \otimes E)^{x^{-1}} \cong H_p(G_{\mathbb{Q}}, E(x)(1))$$

$$u_x \longleftrightarrow [x].$$

Local Restriction:

$$H_p^1(G_{\mathbb{Q}}, E(x)(1)) \longrightarrow H^1(G_p, E(x)(1)) = H^1(G_p, E(1)).$$

Consider

$$\begin{array}{ccc}
 H_p^1(G_{\mathbb{Q}}, E(x^{-1})) & \longrightarrow & H^1(G_p, E(x^{-1})) = H^1(G_p, E) = H_{\text{units}}(\mathbb{Q}_p^{\times}, E) \\
 \text{"} & & = E \cdot \text{ord}_p \oplus E \cdot \log_p
 \end{array}$$

$$\{ [k] \in H^1(G_{\mathbb{Q}}, E(x^{-1})) : \text{res}_{G_l}^{\mathbb{Q}} [k] = 0 \ \forall l \neq p \}$$