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Hida families and Gross-Stark units over totally real fields:

$$\begin{tikzcd} \bar{\mathbb{Q}} \arrow[r, hook] \arrow[dr] & \mathbb{C} \\ & \bar{\mathbb{Q}}_p \end{tikzcd}$$

$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \bar{\mathbb{Q}}^\times \qquad \chi(-1) = \pm 1$

$$\begin{tikzcd} & \mathbb{Q}(\mu_N) \arrow[dl, dash, "\ker\chi"'] \arrow[dd, dash, "(\mathbb{Z}/N\mathbb{Z})^\times"] \\ H \arrow[dr, dash, "\chi"'] & \\ & \mathbb{Q} \end{tikzcd}$$

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1} \quad \text{for } \mathrm{Re}(s) > 1$$

Fix a prime p s.t $\chi(p) = 1$ (i.e, p splits completely in H)

$\chi(a) = 0$ if $\gcd(a, N) \neq 1$.

Let $\omega : G_{\mathbb{Q}} \longrightarrow \mu_{p-1}$ be the Teichmüller character.

p-adic L-function $L_p(\chi\omega, s) : \mathbb{Z}_p \longrightarrow E = \mathbb{Q}_p(\text{values of } \chi) =: \mathbb{Q}_p(\chi)$

s.t. $\quad L_p(\chi\omega, 1-k) = L(\underbrace{\chi\omega^{1-k}}, 1-k) \quad$ for $k \in \mathbb{Z},\ k \geq 1$.

has modulus divisible by p.

Lies in $\mathbb{Q}(\chi, \mu_{p-1})$ $\qquad$ e.g., $\chi\omega^0(p) = 0$, not 1.

$L_p(\chi\omega, 0) = L(\chi\omega^0, 0)$

$L(\chi\omega^0, s) = (1 - \chi(p)p^{-s})\, L(\chi, s)$

$L(\chi\omega^0, 0) = (1 - 1\cdot 1) \cdot L(\chi, 0)$

$= 0.$

<u>Def:</u> $\mathcal{L}_{an}(\chi) = \dfrac{L_p'(\chi\omega, 0)}{L(\chi, 0)} \in E$

$L(\chi, 0) \neq 0$ since $\chi(-1) = -1$

Let $\mathcal{U} = \{u \in H^\times : |u|_w = 1 \ \ \forall\, w \nmid p\}$. This is a finitely
(including arch. primes)

generated abelian group of rank $[H:\mathbb{Q}]/2$.

$\mathcal{U}_\chi = (\mathcal{U} \otimes E)^{\chi^{-1}} = \{u \in \mathcal{U} \otimes E : \sigma(u) = u \otimes \chi^{-1}(\sigma) \ \ \forall\, \sigma \in G_{\mathbb{Q}}\}$.

$\dim_E \mathcal{U}_\chi = 1$. Let u_χ be a generator of $\mathcal{U}_\chi$

$$\begin{tikzcd} & & & \mathbb{Z} \\ \mathcal{U} \subset H^\times \subset \mathbb{Q}_p^\times \arrow[urrr, "\mathrm{ord}_p"] \arrow[drrr, "\log_p"'] \\ & & & \mathbb{Z}_p \end{tikzcd}$$

$(\log_p(p) = 0)$

Tensor with E to obtain:

$$\begin{tikzcd} \mathcal{U} \otimes E \arrow[r, bend left, "\mathrm{ord}_p"] \arrow[r, bend right, "\log_p"'] & E \end{tikzcd}$$

<u>Def:</u> $\mathcal{L}_{alg}(\chi) = -\log_p(u_\chi)/\mathrm{ord}_p(u_\chi)$ $\neq 0$

<u>Thm (Gross):</u> $\mathcal{L}_{an}(\chi) = \mathcal{L}_{alg}(\chi)$.

Gross' proof is explicit, using formula for u_χ in terms of Gauss sums.

Gross-Koblitz formula relates Gauss sums Γ_p, p-adic Γ-function.

Ferrero-Greenberg theorem relates Γ_p to L_p. This gives the proof.
We'll give a different proof using Ribet's method.

One has the same conjecture for totally real fields. However, since there is no explicit CFT here Gross' method does not generalize, which is why this new method is nice.

Ribet's method, Step 1 Reformulation:

Kummer Theory: $\bar{\mathcal{U}} = \{ u \in \bar{\mathbb{Q}}^\times : |u|_w = 1 \text{ for all } w \nmid p \}$

$$1 \longrightarrow \mu_{p^n} \longrightarrow \bar{\mathcal{U}} \xrightarrow{p^n} \bar{\mathcal{U}} \longrightarrow 1$$

Let $\mathcal{O} = \mathbb{Z}[\chi]$. Tensor the short exact sequence by $\mathcal{O}(\chi)$:

$$1 \longrightarrow \mu_{p^n} \otimes \mathcal{O}(\chi) \longrightarrow \bar{\mathcal{U}} \otimes \mathcal{O}(\chi) \xrightarrow{p^n} \bar{\mathcal{U}} \otimes \mathcal{O}(\chi) \longrightarrow 1.$$

Take $G_{\mathbb{Q}}$ - cohomology:

$$0 \longrightarrow (\mathcal{U} \otimes \mathcal{O})^{\chi^{-1}} / p^n \longrightarrow H^1(G_{\mathbb{Q}}, \mu_{p^n} \otimes \mathcal{O}(\chi)) \to H^1(G_{\mathbb{Q}}, \bar{\mathcal{U}} \otimes \mathcal{O}(\chi))[p^n] \to 0.$$

Take $\varprojlim_n$ and then $\otimes_{\mathcal{O}} E$.

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$$\begin{tikzcd}
 & & H_p(G_{\mathbb{Q}}, E(\chi)(1)) \arrow[r] \arrow[d, hook] & \text{“Image”} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathcal{U}\otimes E)^{\chi^{-1}} \arrow[r] \arrow[ur] & H^1(G_{\mathbb{Q}}, E(\chi)(1)) \arrow[r] \arrow[d] & T_p H^1(G_{\mathbb{Q}}, \bar{\mathcal{U}} \otimes \mathcal{O}(\chi)) \otimes_{\mathcal{O}_E} E \arrow[r] \arrow[d, "\alpha"] & 0 \\
 & & \bigoplus_{\ell \neq p} H^1(G_\ell, E(\chi)(1)) \arrow[r] & \bigoplus_{\ell \neq p} T_p(H^1(G_\ell, \bar{\mathcal{U}} \otimes \mathcal{O}(\chi))) \otimes_{\mathcal{O}_E} E \arrow[r] & 0
\end{tikzcd}$$

$$H_p(G_{\mathbb{Q}}, E(\chi)(1)) = \{ [\kappa] \in H^1(G_{\mathbb{Q}}, E(\chi)(1)) : \mathrm{res}^{G_{\mathbb{Q}}}_{G_\ell}[\kappa] = 0 \quad \forall\ \ell \neq p \}$$

$$\text{“Image”} \subseteq \ker\alpha = T_p(\underbrace{\ker(H^1(G_{\mathbb{Q}}) \to \bigoplus_{\ell \neq p} H^1(G_\ell)}_{\text{finite}}) \otimes_{\mathcal{O}_E} E = 0$$

(use inflation-rest. for $G_H \subset G_{\mathbb{Q}}$, use fact that for units, Ш = class group "Neukirch - Schmidt - Winberg" "Cohomology of # fields" Prop 8.3.10(ii))

$$\Rightarrow \quad (\mathcal{U}\otimes E)^{\chi^{-1}} \cong H_p(G_{\mathbb{Q}}, E(\chi)(1))$$

$$u_\chi \longleftrightarrow [\gamma_\chi].$$

Local Restriction:

$$H_p^1(G_{\mathbb{Q}}, E(\chi)(1)) \longrightarrow H^1(G_p, E(\chi)(1)) = H^1(G_p, E(1)).$$

Consider

$$H_p^1(G_{\mathbb{Q}}, E(\chi^{-1})) \longrightarrow H^1(G_p, E(\chi^{-1})) = H^1(G_p, E) = \mathrm{Hom}_{cts}(\mathbb{Q}_p^\times, E)$$

$$= E \cdot \mathrm{ord}_p \oplus E \cdot \log_p$$

$$\| $$

$$\{ [\kappa] \in H^1(G_{\mathbb{Q}}, E(\chi^{-1})) : \mathrm{res}^{G_{\mathbb{Q}}}_{I_\ell}[\kappa] = 0 \quad \forall\ \ell \neq p \}$$