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6-19-10

p-adic Rankin-Selberg convolutions:

Overview: Fix a number field k, some $n \geq 1$, π, σ irred. cuspidal auto. rep. of GL_{n+1}, GL_n over k, assumed of cohomological type.

Fix a finite place $\wp$ of k where π, σ are spherical.

1. Rankin-Selberg L-function following Jacquet, Piatetski-Shapiro, Shalika: $L(s, \pi \times \sigma)$.
2. Explicit formula for $L(\frac{1}{2}, \pi \otimes \chi \otimes \sigma)$ where χ is finite order of $\wp \neq L$. (Assume that π, σ have trivial central character. $s = 1/2$ is central point.)
3. Construction of a distribution/measure s.t.
$$\int \chi \, d\mu = \boxed{\cdots} \, L(\tfrac{1}{2}, \pi \otimes \chi \otimes \sigma).$$
4. Can show algebraicity and boundedness of μ.

Remark: If f is a cusp form of wt k then
$$L(f,s) = L(s - \tfrac{k-1}{2}, \pi(f) \times 1).$$

Exercise: Use our results & Vatsal's lecture to construct $L_p(f,s)$.

1. π, σ cuspidal irred.

$\downarrow$ (Shalika)

$\forall v$ π_v, σ_v are generic

$\rightsquigarrow$ Whittaker models $W(\pi_v, \psi_v) \ni W_v$, $W(\sigma_v, \psi_v^{-1}) \ni v_v$.

$\rightsquigarrow$ local ζ-integral $\rightsquigarrow$ gives local Euler factors (Γ-factor)

$$\left.\begin{array}{l} W = \otimes_v W_v \in W(\pi, \psi) \\ V = \otimes_v V_v \in W(\sigma, \psi^{-1}) \end{array}\right\} \text{Euler product in right half-plane}$$

Fourier transform

$\hookrightarrow$ Analytic continuation:

$$W \longmapsto \phi \in L^2(GL_{n+1}(k) \backslash GL_{n+1}(\mathbb{A}_k))$$
$$V \longmapsto \varphi \in L^2(GL_n(k) \backslash GL_n(\mathbb{A}_k)).$$

2. Local L-functions:

Let v be a finite place of k, $k_v \supseteq \mathcal{O}_v$. Fix an add. char
$\psi_v : k_v \longrightarrow \mathbb{C}^\times$ continuous of exponent 0, $\ker \psi_v = \mathcal{O}_v$.

Write
$$U_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}.$$

We get
$$\psi_v : U_n(k_v) \longrightarrow \mathbb{C}^\times$$
$$(u_{ij})_{ij} \longmapsto \prod_{i=1}^{n-1} \psi_v(u_{i,i+1}).$$

π_v is generic $\iff$ there is a nonzero linear functional
$$\lambda : V_{\pi_v} \longrightarrow \mathbb{C} \quad \text{s.t.}$$
$\forall \; v \in V_{\pi_v}, \forall u \in U_{n+1}(k_v) \qquad \lambda(uv) = \psi_v(u)\lambda(u).$

Also know that λ is unique up to scalars.

Define $v \longmapsto W_v : GL_{n+1}(k_v) \longrightarrow \mathbb{C}$
$$g \longmapsto \lambda(gv).$$

The Whittaker model is given by

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$W(\pi_v, \psi_v) := \{ W_v \; : \; v \in V_{\pi_v} \}$.
$\cap|$

$W_{n+1}(\psi_v) = \{ w : GL_{n+1}(k_v) \to \mathbb{C} \quad \text{s.t. (1), (2) hold} \}$

(1) $\forall u \in U_{n+1}(k_v)$, $\forall g \in GL_{n+1}(k_v)$

$$w(ug) = \psi_v(u) w(g)$$

(2) w is smooth, i.e., $\exists \; K \leq GL_{n+1}(k_v)$ open s.t.

$\forall g$, $\forall k \in K$, $w(gk) = w(g)$.

$\forall \; w \in W(\pi_v, \psi_v)$, $v \in W(\sigma_v, \psi_v^{-1})$.

$$\Psi(s, w \otimes v) := \int_{U_n(k_v) \backslash GL_n(k_v)} w\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} v(g) \, |\det g|_v^{s-1/2} \, dg.$$

$\rightsquigarrow \Psi : W(\pi_v, \psi_v) \times W(\sigma_v, \psi_v^{-1}) \longrightarrow \mathbb{C}(q^{-s}, q^s)$.

These integrals span a fractional ideal $\mathcal{L} \subseteq \mathbb{C}(q^{-s})$ w.r.t. $\mathbb{C}[q^{-s}, q^s]$ (a PID). We have $1 \in \mathcal{L} \Rightarrow$ we can find a generator $T(q^{-s}) = P(q^{-s})^{-1}$ with $P(x) \in \mathbb{C}[x]$ s.t. $P(0) = 1$.

By definition $L(s, \pi_v \times \sigma_v) := P(q^{-s})$. This gives the local theory. Now we do the global theory.

Global L: For any v we find a "good" tensor

$t_v \in W(\pi_v, \psi_v) \otimes W(\sigma_v, \psi_v^{-1})$ s.t. $\Psi(t_v) = L(s, \pi_v \otimes \sigma_v)$.

If π_v, σ_v are spherical then we might choose

$$t_v = \underbrace{w \otimes v}_{\text{essential vectors}}$$

$$t := \otimes_v t_v = \underbrace{\sum_i}_{\text{finite sum}} w_i \otimes v_i \in W(\pi, \psi) \otimes W(\sigma, \psi^{-1})$$

$$\phi_2(g) := \sum_{\gamma \in GL_n(k)} W_2\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

$$\varphi(h) := \sum_{\delta \in GL_{m}(k)} V_2\left(\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} h\right).$$

These are auto. forms.

Re(s) large

$$\int_{GL_n(k)\backslash GL_n(\mathbb{A}_k)} \phi_2\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi_2(g) |\det g|^{s-1/2} \, dg = \int_{U_n(\mathbb{A}_k)\backslash GL_n(\mathbb{A}_k)} W_2\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} V_2(g) \|\det g\|^{s-1/2} \, dg$$

$$= \prod_v \Psi(s, W_{2,v} \otimes V_{2,v}).$$

Taking the sum over 2 gives

$$L(s, \pi \otimes \sigma) = \prod L(s, \pi_v \otimes \sigma_v).$$

<u>Remark:</u> $L(s,\pi\otimes\sigma)$ is entire and satisfies a functional equation of type $s \longleftrightarrow 1-s$.

<u>3. Special values:</u> $\wp$ finite place so that $\pi_\wp, \sigma_\wp$ are spherical. $\varpi \in \wp$ a generator, $f = \varpi^r \neq 1$.

$$h^{(f)} = \begin{pmatrix} f^{-n} & & & & \\ & f^{-m} & & 0 & \\ & & \ddots & & \\ & 0 & & f^{-1} & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 0 & & & 1 \\ & & 0 & \vdots \\ & \iddots & & \vdots \\ 1 & & & \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} f^{n} & & & \\ & f^{n-1} & & 0 \\ & & \ddots & \\ 0 & & f & \\ & & & 1 \end{pmatrix}$$

For any Iwahori invariant $\tilde{\omega}_\wp \in W(\pi_\wp, \psi_\wp)$,

$\tilde{v}_{\mathfrak{p}} \in W(\sigma_{\mathfrak{p}}, \psi_{\mathfrak{p}}^{-1})$, we can construct $\tilde{\phi}_2$, $\tilde{\varphi}_2$ as above.

Thm (Kazdan, Mazur, Schmidt, J.): $\forall\ \chi : k^{\times}\backslash \mathbb{A}_k^{\times} \longrightarrow \mathbb{C}^{\times}$ of conductor $\mathfrak{f}\,\mathcal{O}_{\mathfrak{p}}$ we have the following formula: $\forall s \in \mathbb{C}$

$$\tilde{W}_{\mathfrak{p}}(1)\,\tilde{V}_{\mathfrak{p}}(1) \prod_{v=1}^{n} (1 - N(\mathfrak{p})^{-v})^{-1}\ \tau(\chi)^{\frac{n(n+1)}{2}}\ L(s, \pi \otimes \chi \otimes \sigma) =$$

$$N(\mathfrak{f})^{\sum_{v=1}^{n} v(n+1-v)} \sum_{\ell} \int_{GL_n(k)\backslash GL_n(\mathbb{A}_k)} \tilde{\phi}\left(\begin{pmatrix} g & 0 \\ \ell & 1 \end{pmatrix} h^{(\mathfrak{f})}\right) \tilde{\varphi}(g)\,\chi(\det g)\,|\det g|^{s-1/2}\, dg.$$

Pf: - $\forall\ v \neq \mathfrak{p}$; $\chi_v(\det)\,t_v$ is a good tensor for $L(s, (\pi_v \otimes \chi_v) \otimes \sigma_v)$.

- For $v = \mathfrak{p}$: $L(s, \pi_{\mathfrak{p}} \otimes \psi_{\mathfrak{p}} \otimes \sigma_{\mathfrak{p}}) = 1$ (recall $\psi_{\mathfrak{p}}$ is ramified here!)

We have the following local Birch lemma:

$$\prod_{v=1}^{n} (1 - N(\mathfrak{p})^{-v})^{-1}\ \tau(\chi)^{\frac{n(n+1)}{2}} \int_{U_n(k_{\mathfrak{p}})\backslash GL_n(k_{\mathfrak{p}})} \tilde{w}_{\mathfrak{p}}\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} h^{(\mathfrak{f})}\right) \tilde{v}_{\mathfrak{p}}(g)\,\chi_{\mathfrak{p}}(\det g)\,|\det g|^{s-1/2}\, dg$$

$$= N(\mathfrak{f})^{\sum_{v=1}^{n} v(n+1-v)}\ W_{\mathfrak{p}}(1)\,V_{\mathfrak{p}}(1).$$

From above we know the formula of the theorem holds in a right half-plane (Euler product). By analytic continuation it holds $\forall s \in \mathbb{C}$. ∎

As a corollary of this we get:

Corollary (global Birch lemma): For $s = 1/2$ and χ of finite order, assume $h_{\mathfrak{f}} = 2$ (for simplicity), then the

$$\text{LHS in thm} = N(\mathfrak{f})^{\sum_{v=1}^{n} v(n+1-v)} \sum_{2} \sum_{x \bmod \mathfrak{f}} \chi(x) \int_{C_{x,\mathfrak{f}}} \tilde{\varphi}\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} h^{(\mathfrak{f})}\right) \tilde{\varphi}(g)\, dg \qquad (*)$$

where

$$C_{x,\mathfrak{f}} = \det^{-1}\left(k^{\times} \backslash k^{\times}(x+\mathfrak{f}) \prod_{v \nmid p} \mathcal{U}_v\right)$$

- $\mathcal{U}_v = (k_v^{\times})^{0}$ for $v \mid \infty$
- $\mathcal{U}_v = \mathcal{O}_{k_v}^{\times}$ for $v \nmid \infty$.

Remark: $(*)$ gives 2 results:

- algebraicity of the special values of the twisted L-functions
- a $\mathfrak{p}$-adic distribution interpolating these special values.

4. Parabolic Hecke algebras:

Let G denote a locally compact group, $H \subseteq G$ compact open. $\forall$ subring $A \subseteq \mathbb{C}$ we write

$$\mathcal{H}_A(H, G) = \{ \alpha : G \to A : \alpha \ H\text{-linear} \}$$

This becomes an A-algebra with right convolution:

$$(\alpha * \beta) : x \longmapsto \int_G \alpha(g)\, \beta(xg^{-1})\, dg \qquad \left(\int_H dg = 1 \right).$$

Prop: Let G be locally compact, $K \subseteq G$ compact open, $H \subseteq G$ closed s.t. $HK = G$. Then, $L = H \cap K$

$$\mathcal{H}_A(K, G) \longrightarrow \mathcal{H}_A(L, H)$$
$$\alpha \longmapsto \alpha|_H$$

is a monomorphism of A-algebras.

$G := GL_n(k_{\wp})$, $K = GL_n(\mathcal{O}_{\wp})$, $H = B_n(k_{\wp})$ (standard Borel)
$L = B_n(\mathcal{O}_{\wp})$.

$$B_n(k_{\wp}) \cdot K = G \quad \text{(Iwasawa decomp.)}$$

Therefore we can apply the proposition and get an extension of A-algebras

$$\mathcal{H}_{\wp} := \mathcal{H}(K, G) \subseteq \mathcal{H}(L, H) =: \mathcal{P}_{\wp}.$$

Note: $\mathcal{P}$ is not f.g. and not commutative. But,

$$u_i = 1_{L\begin{pmatrix} 1_{i-1} & & \\ & \varpi & \\ & & 1_{n-i} \end{pmatrix}L} \qquad 1 \leq i \leq n$$

generate over $\mathcal{H}_{\wp}$ a commutative A-algebra $\mathcal{P}' := \mathcal{H}_{\wp}[u_1, \dots, u_n]$ (Gritsenko).