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$\mathfrak{h}_{2n}$ Find F s.t.

$\cup$

$\mathfrak{h}_n \times \mathfrak{h}_n$ $f \otimes f$ appears in $DF|_{\mathfrak{h}_n \times \mathfrak{h}_n}$

ie, want $\langle DF, f \otimes f \rangle_{\mathfrak{h}_n \times \mathfrak{h}_n} \neq 0$.

Key: Doubling method of P.S.-Rallis > analytic,

and Garrett / Shimura > arithmetic.

$V = \mathbb{Q}^{2n}$, $J_n = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$.

$W = V \oplus V$ $\begin{pmatrix} J_n & \\ & -J_n \end{pmatrix}$ ← take − sign to make polarized.

$H = Sp(V)$

$G = Sp(W) \supset H \times H$.

$W = V \oplus V = V_d \oplus V^d$

$V^d = \{(v,v) : v \in V\}$
$V_d = \{(v,-v) : v \in V\}$ > max. isotropic subspaces.

$P = \mathrm{Stab}_G V^d$ (Siegel parabolic)

$\parallel$

MN

$M \xrightarrow{\sim} GL(V^d) \simeq GL(V)$

$P = \begin{pmatrix} {}^tA^{-1} & x \\ & A \end{pmatrix}$, $A \in GL(V^d)$.

$P \cap H \times H = H^d = \{(h,h) \in G : h \in H\}$.

Induced reps and Eisenstein series:

$\chi : \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \longrightarrow \mathbb{C}^{\times}$

$\chi |\cdot|_{\mathbb{A}}^{-s}$ character of M by composing with $\det_M$.

(Modulus character for $P = |\det_M|_{\mathbb{A}}^{-(\frac{n+1}{2})}$)

$$\mathrm{Ind}_P^G \chi|\cdot|_{\mathbb{A}}^{-s} \ni f$$

$$E(f,g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g) \qquad \text{Eisenstein series on } G$$

- converges for $\mathrm{Re}(s) \gg 0$
- mero. cont. in s.
- functional eq. $\quad s \longleftrightarrow -s$

$$f \longleftrightarrow \underbrace{M(f,s)}_{\text{intertwining operator.}}$$

Doubling integral:

π cusp. auto. rep. of $H/\mathbb{A}$.

$\varphi_1 \in \pi$, $\varphi_2 \in \tilde{\pi}$.

$$I(\chi,\varphi_1,\varphi_2,f) = \int_{(H\times H)_{\mathbb{Q}}\backslash (H\times H)_{\mathbb{A}}} E(f,(g_1,g_2))\,\varphi_1(g_1)\varphi_2(g_2)\,dg_1 dg_2.$$

$$= \underbrace{\sum_{\gamma \in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/(H\times H)_{\mathbb{Q}}}}_{\text{finite}} \int_{H^\gamma\backslash (H\times H)_{\mathbb{A}}} f(g_1,g_2)\,\varphi_1(g_1)\varphi_2(g_2)\,dg_1 dg_2.$$

$$H^\gamma = (H\times H)_{\mathbb{Q}} \cap \gamma^{-1} P_{\mathbb{Q}} \gamma.$$

($\gamma \neq 1$ $\quad H^\gamma$ has a normal subgroup that is a unipotent radical of a parabolic of $H\times H$. $\Rightarrow$ integral vanishes)

$$= \int_{H^d_{\mathbb{Q}}\backslash (H\times H)_{\mathbb{A}}} f(g_1,g_2)\,\varphi_1(g_1)\varphi_2(g_2)\,dg_1 dg_2$$

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$$\left(H \times H = H^d (H \times H) \right)$$

$$= \int_{H_{\mathbb{A}}} f(g,1) \Big(\underbrace{\int_{H_{\mathbb{Q}} \backslash H_{\mathbb{A}}} \varphi_1(g'g)\,\varphi_2(g')\, dg'}_{\langle \pi(g)\varphi_1, \varphi_2 \rangle} \Big)\, dg$$

$$= \int_{H_{\mathbb{A}}} f(g,1) \langle \pi(g)\varphi_1, \varphi_2 \rangle \, dg.$$

If $f = \prod f_v$, $\varphi_i = \otimes \varphi_{i,v} \in \otimes \pi_{i,v} = \pi_i$ $\quad \pi_1 = \pi$, $\pi_2 = \tilde{\pi}$

$\langle \, , \rangle = \prod \langle , \rangle_v$.

Then we have

$$I(\chi, \varphi_1, \varphi_2, f) = \prod_v \underbrace{I(\chi_v, \varphi_{1,v}, \varphi_{2,v}, f_v)}$$

$$= \int_{H_v} f_v((g,1)) \langle \pi_v(g)\varphi_{1,v}, \varphi_{2,v} \rangle_v \, dg$$

<u>Unramified calculation:</u> (P.S.-Rallis)

If $\chi_v, \pi_v, f_v, \varphi_{i,v}$ are unram., then

$$I(\chi_v, \varphi_{1,v}, \varphi_{2,v}, f_v) = \frac{L(\pi_v \times \chi_v, s + 1/2)}{d(\chi_v, s)}.$$

π_v = principal series induced from a character of the torus $\left\{ \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_1^{-1} & & \\ & & & & \ddots & \\ & & & & & x_n^{-1} \end{pmatrix} \right\} \longleftarrow \mu_1, \dots, \mu_n$

$$L(\pi_v \times \chi_v, s+\tfrac{1}{2}) = L(\chi_v, s) \prod_{i=1}^{n} (1-\mu_i \chi_v(\ell)\ell^{-s})^{-1}(1-\mu_i^{-1}\chi_v(\ell)\ell^{-s})^{-1}.$$

$$d(\chi_v, s) = L(\chi_v, s) \prod_{\substack{j=2 \\ j\equiv 0 \,(\mathrm{mod}\, 2)}}^{n} L(2s-j, \chi_v^2).$$

$$I = \frac{L^{\Sigma}(\pi\times\chi, s+\tfrac{1}{2})}{d^{\Sigma}(\chi, s)} \prod_{v\in\Sigma} I_v,$$

Σ = finite set

doubling integral

<u>Remark:</u> Can choose the $\varphi_{i,v}$, f_v at $v\in\Sigma$ so that $I_v \neq 0$.

For applications to families, one must be more careful/precise for $v\in\Sigma$. This can be done as well.

$$Sp(V)\times Sp(V) \hookrightarrow Sp(W) \xrightarrow{\ \sim\ } Sp\left(V_d \oplus V^d, \begin{pmatrix} & 1_{2n} \\ -1_{2n} & \end{pmatrix}\right)$$

$$Sp_{2n}\times Sp_{2n} \qquad \begin{pmatrix} J & \\ & J \end{pmatrix}$$

Geometrically, A_i = abelian variety, then we are looking at:

$$(A_i, \varphi_i, \mu_{p^n} \hookrightarrow A[p]) \longrightarrow (A_1\times A_2, \varphi_1\times\varphi_2)$$

$$\mathfrak{H}_n \times \mathfrak{H}_n \hookrightarrow \mathfrak{H}_{2n}$$

$$1, \tau_1, 1, \tau_2 \qquad \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}.$$

$$E(f,g) \longrightarrow E(Z)$$

$$Z \in \mathfrak{H}_{2n}$$

$E(Z)$ has a Fourier expansion with coefficients given by:

$$\int_{\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\in N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} E\left(\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)g\right)\psi(-\mathrm{tr}\,\beta x)\,dx \qquad \begin{matrix}x={}^tx\\ \beta={}^t\beta\end{matrix}$$

$$\shortparallel$$

$$E_\beta(1)$$

If some fn is supported on $P\underbrace{\left(\begin{smallmatrix}&1_n\\-1_n&\end{smallmatrix}\right)}_{w}P$ then

$$E_\beta(1)=\prod_v\int_{\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\in N_{\mathbb{A}}} f\left(w\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\right)\psi_v(-\mathrm{tr}\,\beta x)\,dx$$

Unramified calculations (Shimura) gives a nice formula.
He also does arch. calculations (Shimura)

$v\in\Sigma$? Can deal with $v\in\Sigma$, $v\neq p$ can be dealt w/ by choosing sections appropriately.

Problem is at $v=p$.

Siegel (-Weyl) sections:

$\Phi\in\mathcal{S}(M_{2n\times 4n})$ Schwartz functions
$\qquad\shortparallel$
$\quad M(V)\oplus M(V)$

$$f^{\Phi}(g)=\int_{GL_n(V)}\Phi(\underbrace{(h,h)}_{V^d}g)\,\chi^{-1}(\det h)\,|\det h|^s\,dh.$$

$\in \mathrm{Ind}\,\chi_v|\cdot|_v^{-s}$.

$\Phi'((0,h)\gamma(g))$.

$h\in GL(V^d)$

$\Phi'(x,y)=\Phi_1(x)\Phi_2(y)$

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local contribution to f.c.

$$\int_X \int_{GL(V)} \Phi_1(h)\,\Phi_2(xh)\,\psi(-\mathrm{tr}(\beta x))\,dx\,dh$$

For good choices of Φ_1, Φ_2 get $\hat{\Phi}_2(\beta)$.

Can now reverse engineer this to figure out what the section at p should be. This is fine for the f.c. One then still must do the calculation of I_p. (This is calculations of J.S. Li, Harris, ~~~~ S.)