Vatsal

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PS1

$f \in S_2(\Gamma_1(N))^{new}$ $(N > 4)$

<u>idea:</u> produce periods $\Omega_f^\pm$ s.t. $\dfrac{L(f,\eta,1)}{2\pi\Omega_f^\pm}$ is algebraic where $\pm$ is determined by $\eta(-1)$.

Consider $\omega_f = f(z)dz \in H^1(X_1(N),\mathbb{C})$. Project to the $\pm$ spaces:

$$\omega_f^\pm \in H^1(X,\mathbb{C})^\pm.$$

$\mathbb{T}$ = hecke algebra, then the kernel $\wp_f$ (= prime ideal corresponding to $T_\ell \longmapsto a_\ell(f)$) has dim 1 over $\mathbb{C}$. $\omega_f^\pm \in H^1(X,\mathbb{C})^\pm[\wp_f]$.

$$\overset{\shortparallel}{H^1(X,\mathbb{Q})^\pm[\wp_f]\otimes\mathbb{C}}$$ has dim 1 over $\mathbb{T}/\wp_f = K_f$.

Pick $\delta_f^\pm \in H^1(X,\mathbb{Q})^\pm[\wp_f]$ generating it as a $K_f$-space. So $\Omega_f^\pm\,\delta_f^\pm = \omega_f^\pm$.

If $\gamma$ is any cycle in $H_1(X,\mathbb{Q})$, then we get $\omega_f\cdot\gamma = \Omega_f^\pm\,\underbrace{\delta_f^\pm\cdot\gamma}$

algebraic since $\delta_f^\pm$ is algebraic cycle.

$$\Rightarrow \quad \frac{\omega_f^\pm\cdot\gamma}{\Omega_f^\pm} \in \overline{\mathbb{Q}}.$$

<u>Point:</u> Want to pick $\gamma = \gamma(\eta)$ to be algebraic over $\mathbb{Q}(\eta)$ and

s.t. $\omega_f^\pm\cdot\gamma(\eta) = \dfrac{L(f,\eta,1)}{2\pi}\,\tau(\overline{\eta})$

$\gamma(\eta)$ is defined as a relative class relative to cusps: $M = \text{cond}(\eta)$

$$\sum_{a=1}^{M}\eta(a)\left\{\frac{a}{M},\infty\right\}$$

Manin-Drinfeld: This is actually a rational class.

Relative homology sequence:

$$\begin{tikzcd}
0 \arrow[r] & H_1(X, \mathbb{Q}) \arrow[r] & H_1(X, c; \mathbb{Q}) \arrow[r] & \widetilde{H}_0(c, \mathbb{Q}) \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
& \{\frac{a}{M}, \infty\} \arrow[ur] & & \mathrm{Div}^0(\text{cusps}) &
\end{tikzcd}$$

Manin-Drinfeld: this sequence has a canonical splitting $j$

$$\{\tfrac{a}{M}, \infty\} = [\tfrac{a}{M}, \infty] + j(\tfrac{a}{M} - \infty) \quad \text{in } H_1(X, \mathbb{Q})$$

Canonically and $\omega_f^{\pm} \cdot [\frac{a}{M}, \infty] = \int_{\frac{a}{M}}^{\infty} \omega_f^{\pm}$.

Idea: use the Hecke action. Pick a prime $p \equiv 1 \pmod{N}$.

Consider $T_p - (p+1) = \widetilde{T}_p$ acting on $H^1(X, c; \mathbb{Q})$. It is clear that $\widetilde{T}_p$ acts invertibly on $H_1(X, \mathbb{Q})$. A computation shows it kills the quotient $\mathrm{Div}^0(C)$. The splitting is effected by the kernel of this operator.

$$H^1(X, c; \mathbb{Q}) = H_1(X, \mathbb{Q}) \oplus j(\mathrm{Div}^0(\text{cusps}))$$

$$([\tfrac{a}{M}, i\infty],\ j(\tfrac{a}{M} - \infty))$$

Not a priori clear: how to canonically pair $H_1(X, C; \mathbb{Q})$ with $\omega_f \in H^1(X, \mathbb{C})$.

The operation $\omega_f \longmapsto \int_a^{\infty} \omega_f$ $\quad a = \text{cusp}$

corresponds to a map

$$\begin{tikzcd}
\omega_f \in H^1(X, \mathbb{C}) \arrow[r] & H^1(X, c; \mathbb{C})^{\vee} \arrow[d, phantom, "\parallel\ \text{Lefschetz duality}"] \\
& H^1(Y, \mathbb{C})^{\vee} \simeq H^1_c(Y, \mathbb{C})
\end{tikzcd}$$

There is a map

$$\begin{tikzcd} H^1_c(Y, \mathbb{C}) \arrow[rr] \arrow[dr] & & H^1(Y, \mathbb{C}) \\ & H^1(X, \mathbb{C}) \arrow[ur] & \end{tikzcd}$$

Write $\tilde{\omega}_f$ for the lift of $\omega_f$ to $H^1_c(Y, \mathbb{C})$.

WANT: $\omega_f([\frac{a}{M}, \infty]) = \tilde{\omega}_f$ on $\{\frac{a}{M}, \infty\}$. The proof of this is an exercise.

Reason for digression: $\{\frac{a}{M}, \infty\}$ are not in $H_1(X, \mathbb{Q})$.

Notice that $\gamma(\eta)$ often are genuine cycles.

e.g. when dealing with $\Gamma_0(N)$ and $\eta$ of cond. prime to N.

Check that $\partial(\gamma(\eta)) = 0$. This means don't have to worry about the splitting.

For higher weight: easiest to work with group cohomology

$$f \rightsquigarrow \text{get a cocycle } \omega_f \in H^1(\Gamma_1(N), \mathbb{C})$$

$$g \in \Gamma \longmapsto \int_{z_0}^{g z_0} f(z)\,dz.$$

Fact: $\omega_f \in H^1_{par}(\Gamma_1(N), \mathbb{C})$

$$\shortparallel$$

$$\bigcap_{\substack{s \text{ a cusp} \\ \text{of } \Gamma_1(N)}} \operatorname{Ker}\{\operatorname{Res}(\Gamma_1(N) \to \underset{\substack{\shortparallel \\ \mathrm{Stab}_s \le \Gamma_1(N)}}{\Gamma_s})\}$$

$$\begin{array}{ccc} H^1(\Gamma_1(N),\mathbb{C}) & \cong & H^1(Y,\mathbb{C}) \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ H^1_{par}(\Gamma_1(N),\mathbb{C}) & = & \operatorname{im} H^1_c(Y,\mathbb{C}) \\ & & \| \\ & & \operatorname{im} H^1(X,\mathbb{C}). \end{array}$$

For wt $k \geq 2$, introduce a module $V_k = \mathrm{Sym}^k(\mathbb{C}^2)$, i.e., the homogeneous polys. of deg $n = k-2$ with the obvious action of $SL_2(\mathbb{Z})$. This is a $\Gamma$-module.

<u>Thm (Eichler-Shimura):</u>

$$H^1_{par}(\Gamma_1(N),V_k)^{\pm} \cong S_k(\Gamma_1(N))$$ as a Hecke module.

<u>Also</u> $$H^1_{par}(\Gamma_1(N),V_k) \cong \underset{\substack{\uparrow \\ \text{stable under Hecke.}}}{H^1_{par}(\Gamma_1(N),V_k(\mathbb{Q}))}\otimes\mathbb{C}$$

<u>Question:</u> How to evaluate $\omega_f^{\pm}$ on $\{\frac{a}{M},\infty\}$?

The calculation from the 1st lecture suggests there will be a relation with L-values.

<u>Construction of a complex cocycle:</u>

$$g \longmapsto \int_{z_0}^{gz_0} f(z)\,(X-zY)^n\,dz.$$

This is a cocycle in $H^1(\Gamma_1(N),V_k)$, and turns out to be parabolic.

Same picture as before:

$$\begin{tikzcd} & H^1(\Gamma_1(N), V_k) \arrow[d, equal] \\ H^1_c(Y, \tilde{V}_k) \arrow[r] & H^1(Y, \tilde{V}_k) \end{tikzcd}$$

$\mathrm{im}\, H^1_c = H^1_{par}$,

The cocycle $\omega_f$ lifts to a compactly supported class.

$$H^1_c(Y, V_k) \underset{\substack{\uparrow \\ \text{Ash-Stevens, works over } \mathbb{Q} \text{ as well}}}{\cong} \mathrm{Hom}_{\Gamma}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), V_k)$$

Evaluate compactly supported class on $\{\frac{a}{M}, \infty\}$ in $\mathrm{Div}^0$

The arg. is the same.

Warning: The lift from $H^1_{par}(\Gamma_1(N), -)$ to $H^1_c(Y, -)$ is supposed to be effected by $\int_{\frac{a}{M}}^{\infty} -$. Not clear (to me) this is compatible with the various identifications made along the way.