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Mixed-level Saito–Kurokawa liftings

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Abstract In this article we construct Saito–Kurokawa lifts of mixed level. These are constructed via representation theoretic arguments originally used by Schmidt to construct congruence level and paramodular Saito–Kurokawa lifts.

Keywords Saito–Kurokawa lifts · Siegel modular forms · Modular correspondences

Mathematics Subject Classification Primary 11F32 · Secondary 11F12 · 11F46

1 Introduction

From a classical point of view, the Saito–Kurokawa lifting of full level has been known for some time due to the work of many mathematicians culminating in the work of Zagier [17]. This lifting provides a way to associate to a newform $f \in S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z}))$ with $\kappa \geq 2$ even a Siegel eigenform $F_f \in S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ satisfying

$$L(s, F_f, \mathrm{spin}) = \zeta(s-\kappa+1)\zeta(s-\kappa+2)L(s, f).$$

It is natural to ask if such lifts exist for forms with level $\Gamma \subsetneq \mathrm{SL}_2(\mathbb{Z})$. There are two classical constructions that generalize the full-level lifting.

Combining the work of Gritsenko [5] and Skoruppa–Zagier [15], one can generalize the full level Saito–Kurokawa lifting to paramodular levels, namely, given a newform

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$f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(m))$ with the sign of the functional equation of f being -1, there is a paramodular Siegel modular form $F_f \in S_\kappa(\Gamma[m])$ satisfying the same L-function relationship as in the full-level case. The Saito–Kurokawa lifting with square-free congruence level was claimed in [9]. Although there was an error in the definition of the Maass lifting with level in [9] and the lifting was never proved to be cuspidal, one does have a congruence square-free level Saito–Kurokawa lifting [2,6]. This associates $F_f \in S_\kappa(\Gamma_0^{(2)}(m))$ to a newform $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(m))$ with $\kappa \geq 2$ an even integer and m square-free satisfying

$$L(s, F_f, \text{spin}) = \left(\prod_{\substack{p|m \\ \epsilon_p=-1}} (1 - p^{-s+\kappa-1}) \right) \zeta(s-\kappa+1)\zeta(s-\kappa+2)L(s, f).$$

In the case $m = 1$, the paramodular- and congruence-level Saito–Kurokawa liftings both specialize back to the full-level case. The first proof of the cuspidality of the Maass lifting with congruence level was given in [6]. The second author provided a proof of the cuspidality of both the congruence level and the paramodular level liftings in [18] using the cusp structure of $\Gamma \backslash \text{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$, where $C_{2,1}(\mathbb{Q})$ is a maximal parabolic subgroup and Γ is $\Gamma_0^{(2)}(M)$ or $\Gamma[m]$.

Each of these lifts is useful in applications to problems in arithmetic. For example, one can see [1,7,14] for such applications. As such, a unified approach to these lifts is desirable. Using the language of representation theory, Schmidt [12,13] was able to provide this unification. In this paper we modify his construction to obtain more general Saito–Kurokawa lifts of mixed level. It should be noted that given a newform $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(m))$, there are generally many Saito–Kurokawa lifts of f with various levels, see Theorem 3.4. From a classical point of view, the elliptic-Jacobi map needed for such mixed-level lifts was claimed in [8]. However, very few proofs are provided and the known errors from [9] are not addressed. The cuspidal Jacobi–Siegel map of the mixed-level lifts from a classical point of view can be found in [18]. The entire mixed-level lifting from a classical point of view is the ongoing work of the second author. In this paper we provide a representation theoretic construction of such mixed-level lifts.

This paper is organized as follows: In Sect. 2 we gather notation and definitions needed for the rest of the paper. We recall Schmidt's results and modify his arguments to provide more general Saito–Kurokawa lifts in Sect. 3. In particular, Theorem 3.4 is the main result of the paper. Finally, in Sect. 4 we specialize Theorem 3.4 to recover the mixed-level lifts claimed in [8].

2 Notation

In this section we collect notation that will be used in the rest of the paper. Throughout this paper we let m be a positive square-free integer.

2.1 Elliptic modular forms

We let GL_2 and SL_2 have their standard definitions. We write $\mathrm{GL}_2^+(\mathbb{R})$ for the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of matrices with positive determinant. Given a prime p, we will make use of the group

$$K_{0,p}(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : c \equiv 0 \bmod p\mathbb{Z}_p \right\}.$$

Let m be a positive square-free integer. We define

$$K_0(m) = \prod_{p \mid m} K_{0,p}(p) \prod_{p \nmid m} \mathrm{GL}_2(\mathbb{Z}_p).$$

Set $\Gamma_0(m) = \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{GL}_2^+(\mathbb{R}) K_0(m)$.

Given an integer $\kappa \geq 2$, we let $S_\kappa(\Gamma_0(m))$ denote the elliptic cusp forms of weight κ and level $\Gamma_0(m)$ and $S_\kappa^{\mathrm{new}}(\Gamma_0(m))$ the subspace of new forms.

Let $f \in S_\kappa^{\mathrm{new}}(\Gamma_0(m))$ be a newform. We denote the cuspidal automorphic representation associated to f by $\pi_f = \otimes \pi_{f,p}$. Recall that $\pi_{f,\infty}$ is the discrete series representation with lowest weight vector of weight κ and, for $p \nmid m$, we have $\pi_{f,p}$ as the unramified principal series representation. The local representations for $p \mid m$ are determined by the Atkin–Lehner eigenvalues of f. For $p \mid m$, recall that the Atkin–Lehner operator at p is given by the matrix $W_p = \begin{pmatrix} pa & b \\ mc & pd \end{pmatrix}$, where a, b, c, d are integers such that $p^2 ad - mbc = p$. If $f \in S_\kappa^{\mathrm{new}}(\Gamma_0(m))$, we let $\epsilon_p \in \{\pm 1\}$ denote the Atkin–Lehner eigenvalue of f at p, i.e., $W_p f = \epsilon_p f$. If $\epsilon_p = -1$, then $\pi_{f,p} = \mathrm{St}_{\mathrm{GL}(2)}$ and if $\epsilon_p = 1$ then $\pi_{f,p} = \xi\, \mathrm{St}_{\mathrm{GL}(2)}$, where $\mathrm{St}_{\mathrm{GL}(2)}$ is the Steinberg representation and ξ is the unique non-trivial unramified quadratic character of $\mathbb{Q}_p^\times$.

We will also need L-functions attached to f and π_f. For each prime $p \nmid m$, there exists a character σ_p so that $\pi_{f,p} = \pi(\sigma_p, \sigma_p^{-1})$ (see [4, Sect. 4.5]). The p-Satake parameter of f is given by $\alpha_0(p; f) = \sigma_p(p)$. We have

$$L(s, \pi_{f,p}) = (1 - \alpha_0(p; f) p^{-s})^{-1} (1 - \alpha_0(p; f)^{-1} p^{-s})^{-1}.$$

For $p \mid m$ we have

$$L(s, \pi_{f,p}) = (1 + \epsilon_p p^{-s-1/2})^{-1}$$

and for $p = \infty$ we set

$$L(s, \pi_{f,\infty}) = (2\pi)^{-(s+(k-1)/2)} \Gamma(s + (k-1)/2).$$

Set

$$L(s, \pi_f) = \prod_p L(s, \pi_{f,p}).$$

The functional equation for $L(s, \pi_f)$ is given by

$$L(s, \pi_f) = \epsilon(s, \pi_f) L(1 - s, \pi_f),$$

where $\epsilon(s, \pi_f) = \prod_p \epsilon(s, \pi_{f,p})$ and

$$\epsilon_p(s, \pi_{f,p}) = \begin{cases} (-1)^{-\kappa/2} & \text{if } \; p = \infty, \\ -p^{1/2-s} & \text{if } \; \epsilon_p = -1, \, p < \infty, \\ p^{1/2-s} & \text{if } \; \epsilon_p = 1, \, p < \infty. \end{cases}$$

In particular, the sign of the functional equation is given by $\epsilon(1/2, \pi_f) \in \{\pm 1\}$. We will phrase our main results in terms of the L-function associated to f. One has

$$L(s, f) = \prod_{p<\infty} L(s + 1/2 - \kappa/2, \pi_{f,p}).$$

2.2 Siegel modular forms

Let GSp_4 be the symplectic group realized via the symplectic form $J = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$. We write $\mathrm{GSp}_4^+(\mathbb{R})$ to denote the subgroup of $\mathrm{GSp}_4(\mathbb{R})$ consisting of matrices with positive determinant.

For a prime p define

$$K_{0,p}^{(2)}(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{Z}_p) : c \equiv 0 \bmod p\mathbb{Z}_p \right\}$$

and

$$K_p^{(2)}[p] = \left\{ \gamma = \begin{pmatrix} a_1 & pa_2 & b_1 & b_3 \\ a_3 & a_4 & b_3 & p^{-1}b_4 \\ c_1 & pc_2 & d_1 & d_2 \\ pc_3 & pc_4 & pd_3 & d_4 \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{Q}_p) : a_i, b_i, c_i, d_i \in \mathbb{Z}_p, \quad \det(\gamma) \in \mathbb{Z}_p^\times \right\}.$$

Given a positive square-free integer m, set

$$K_0^{(2)}(m) = \prod_{p|m} K_{0,p}^{(2)}(p) \prod_{p\nmid m} \mathrm{GSp}_4(\mathbb{Z}_p)$$

and

$$K^{(2)}[m] = \prod_{p|m} K_p^{(2)}[p] \prod_{p\nmid m} \mathrm{GSp}_4(\mathbb{Z}_p).$$

Let $\Gamma \subset \mathrm{Sp}_4(\mathbb{Q})$ be a subgroup commensurable with $\mathrm{Sp}_4(\mathbb{Z})$. We let $S_\kappa(\Gamma)$ denote the space of Siegel modular forms of weight κ and level Γ. Given $F \in S_\kappa(\Gamma)$, F generates a space of cuspidal automorphic forms on $\mathrm{GSp}\,4(\mathbb{A})$ invariant under right translation. In general, this space may not be irreducible, but does decompose into a finite number of irreducible, cuspidal, automorphic representations. Let Π_F be one of these irreducible pieces; we can decompose it into local components $\Pi_F = \otimes \Pi_{F,p}$ with $\Pi_{F,p}$, a representation of $\mathrm{PGSp}_4(\mathbb{Q}_p)$. We refer the reader to [3, Sect. 3] for details concerning the construction of cuspidal automorphic representations associated to Siegel cusp forms.

Given $F \in S_\kappa(\Gamma)$ as above, for all but finitely many places p the representation $\Pi_{F,p}$ will be an Iwahori-spherical representation $\Pi(\sigma, \chi_1, \chi_2)$, which is isomorphic to the Langlands quotient of an induced representation of the form $\chi_1 \times \chi_2 \rtimes \sigma$ with χ_i and σ unramified characters of $\mathbb{Q}_p^\times$. One can see [3,11] for the definitions and details. For such p, the p-Satake parameters are defined by $b_0 = \sigma(p)$ and $b_i = \chi_i(p)$ for $i = 1, 2$. We define

$$L(s, \Pi_{F,p}, \mathrm{spin}) = \left((1 - b_0 p^{-s})(1 - b_0 b_1 p^{-s})(1 - b_0 b_2 p^{-s})(1 - b_0 b_1 b_2 p^{-s})\right)^{-1}$$

for $\Pi_{F,p} = \Pi(\sigma, \chi_1, \chi_2)$. We leave the local L-functions for those p where $\Pi_{F,p}$ is not of the form $\Pi(\sigma, \chi_1, \chi_2)$ and the L-function at the infinite prime undefined for now as these will be given in the next section. Set

$$L(s, \Pi_F, \mathrm{spin}) = \prod_p L(s, \Pi_{F,p}, \mathrm{spin}).$$

As in the GL_2 case, we will phrase our results in terms of $L(s, F, \mathrm{spin})$. The relation is given by

$$L(s, F, \mathrm{spin}) = L(s - \kappa + 3/2, \Pi_F, \mathrm{spin}).$$

3 Saito–Kurokawa lifts

Working within the framework of Langlands' functoriality, Schmidt [12] unified both constructions of the Saito–Kurokawa lifting described in Sect. 1. In this section we recall his results and show how his methods can be used to give more general Saito–Kurokawa liftings.

3.1 Schmidt's Main Lifting Theorem

Let $\mathcal{S}$ be a finite set of places of $\mathbb{Q}$. As in [12], we let $\pi_{\mathcal{S}} = \otimes \pi_{\mathcal{S},p}$ be the non-cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ so that

$$\pi_{\mathcal{S},p} = \begin{cases} \mathbf{1}_{\mathrm{GL}_2} & \text{if } p \notin \mathcal{S} \\ \mathrm{St}_{\mathrm{GL}_2} & \text{if } p \in \mathcal{S}, \end{cases}$$

where $\mathbf{1}_{\mathrm{GL}_2}$ is the trivial representation and $\mathrm{St}_{\mathrm{GL}_2}$ is the Steinberg representation for PGL_2.

We now state Schmidt's main lifting theorem.

Theorem 3.1 ([12], Theorem 3.1) *Let m be a positive square-free integer, $f \in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(m))$ a newform, and $\pi_f = \otimes_p \pi_{f,p}$ be the cuspidal automorphic representation associated to f. Let $\mathcal{S} \subset \{p \mid m\} \cup \{\infty\}$ so that $(-1)^{\#S} = \epsilon(1/2, \pi_f)$. Let $\pi_{\mathcal{S}} = \otimes \pi_{\mathcal{S},p}$ be the non-cuspidal automorphic representation of* $\mathrm{PGL}_2(\mathbb{A})$ *as defined above. Then*

(1) *The global lifting $\Pi(\pi_f \otimes \pi_{\mathcal{S}}) := \otimes \Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p})$ is an automorphic representation of* $\mathrm{PGSp}_4(\mathbb{A})$ *which appears discretely in the space of automorphic forms.*
(2) *If $L(1/2, \pi_f) = 0$ or if $\mathcal{S} \neq \emptyset$, then $\Pi(\pi_f \otimes \pi_{\mathcal{S}})$ is a cuspidal automorphic representation.*

3.2 Local components of $\Pi(\pi_f \otimes \pi_{\mathcal{S}})$

We now describe the local representations $\Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p})$ as is done in [13, Sect. 5].

We note that since we are ultimately interested in holomorphic Siegel modular forms, we will require our $\mathcal{S}$ to always contain ∞. Therefore when it exists, the global lifting will be a cuspidal automorphic representation of $\mathrm{PGSp}_4(\mathbb{A})$ whose local component at ∞ is given by $\Pi(\pi_{f,\infty} \otimes \mathrm{St}_{\mathrm{GL}(2)})$. If we assume f has weight $2\kappa - 2$, the archimedean component $\pi_{f,\infty}$ is the discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ with a lowest weight vector $2\kappa - 2$ denoted by $D(2\kappa - 3)$. The archimedean component $\pi_{\mathcal{S},\infty} = \mathrm{St}_{\mathrm{GL}(2)}$ is the lowest discrete series representation $D(1)$ of $\mathrm{PGL}_2(\mathbb{R})$. It is shown in [3] that $\Pi(D(2\kappa - 3) \otimes D(1))$ is the archimedean component of an automorphic representation corresponding to a holomorphic Siegel modular form of weight κ. In other words, $\Pi(D(2\kappa - 3) \otimes D(1))$ is the holomorphic discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with weight κ.

For places $p \nmid m$, the local representations $\Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p})$ are unramified of the form $\Pi(\pi_{f,p} \otimes \mathbf{1}_{\mathrm{GL}_2})$, where $\pi_{f,p}$ is the unramified principal series representation containing non-zero vectors fixed by $\mathrm{GL}_2(\mathbb{Z}_p)$.

It remains to describe the local representations of the global lifting for finite places $p \mid m$. Recall for $p \mid m$ we described the local representations $\pi_{f,p}$ in terms of the Atkin–Lehner eigenvalues ϵ_p of f in the previous section. Consequently, there is a total of four possibilities for $\Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p})$:

(1) $\Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}_{\mathrm{GL}(2)})$ if $p \notin \mathcal{S}$ and $\epsilon_p = -1$,
(2) $\Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)})$ if $p \in \mathcal{S}$ and $\epsilon_p = -1$,
(3) $\Pi(\xi\, \mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}_{\mathrm{GL}(2)})$ if $p \notin \mathcal{S}$ and $\epsilon_p = 1$,
(4) $\Pi(\xi\, \mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)})$ if $p \in \mathcal{S}$ and $\epsilon_p = 1$.

The fourth representation is known to be supercuspidal ([12, Summary p. 24]), so it has no Iwahori-fixed vectors. As such, it will play no role in our work as it cannot be the component of a, the representation arising from a classical holomorphic Siegel modular form. Thus, if $p \in \mathcal{S}$ is a finite prime, we will require $\epsilon_p = -1$.

Recall that the local Atkin–Lehner involution at p is given by

$$\eta_p = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

The Atkin–Lehner eigenvalue of $\Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}_{\mathrm{GL}(2)})$ is -1 and the Atkin–Lehner eigenvalues of $\Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)})$ and $\Pi(\xi\, \mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}_{\mathrm{GL}(2)})$ are equal to 1 ([13, Table 30]).

We will also have the use of the local L-factors of the local representations $\Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p})$ for those $p \mid m$. These are given in [13, Table 33]:

$$\begin{aligned} L(s, \Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}_{\mathrm{GL}(2)})) &= (1 - p^{-s-1/2})^{-2}(1 - p^{-s+1/2})^{-1}, \\ L(s, \Pi(\xi\, \mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}_{\mathrm{GL}(2)})) &= (1 - p^{-s-1/2})^{-1}(1 - p^{-s+1/2})^{-1}(1 + p^{-s-1/2})^{-1}, \\ L(s, \Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)})) &= (1 - p^{-s-1/2})^{-2}. \end{aligned}$$

3.3 The relationship to classical Saito–Kurokawa lifts

The main result of [13] is the following theorem that constructs the classical and paramodular Saito–Kurokawa lifts via Theorem 3.1.

Theorem 3.2 [[13], Theorem 5.2] *Let m be a square-free positive integer and $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(m))$ a newform. Let ϵ_p be the eigenvalue of f of the Atkin–Lehner involution W_p at p. Let η_p be the Atkin–Lehner involution at p as defined above.*

(1) *If $\epsilon(1/2, \pi_f) = -1$, then there exists a cusp form $F_f \in S_\kappa(\Gamma[m])$, unique up to scalar multiples, satisfying*

$$L(s, F_f, \mathrm{spin}) = \zeta(s - \kappa + 1)\zeta(s - \kappa + 2)L(s, f).$$

The lifting preserves Atkin–Lehner eigenvalues, i.e., the Atkin–Lehner eigenvalue of $\Pi_{F_f,p}$ under η_p is ϵ_p.

(2) *If κ is even, then there exists a cusp form $F_f \in S_\kappa(\Gamma_0(m))$, unique up to scalar multiples, satisfying*

$$L(s, F_f, \mathrm{spin}) = \left(\prod_{\substack{p \mid m \\ \epsilon_p = -1}} (1 - p^{-s+\kappa-1}) \right) \zeta(s - \kappa + 1)\zeta(s - \kappa + 2)L(s, f).$$

The Atkin–Lehner eigenvalue of $\Pi_{F_f,p}$ is 1 for each $p \mid m$.

In fact, one can use Theorem 3.1 to construct more general Saito–Kurokawa lifts. Let $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(m))$ be a newform. Set $\mathcal{T} = \{p \mid m : \epsilon_p = -1\}$ and $\mathcal{W} = \{p \mid m : \epsilon_p = 1\}$.

Definition 3.3 Let $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(m))$ be a newform. Let $\mathcal{S} \subset \mathcal{T} \cup \{\infty\}$ so that $\infty \in \mathcal{S}$ and $(-1)^{\#\mathcal{S}} = \epsilon(1/2, \pi_f)$. We will call such a set $\mathcal{S}$ *an admissible set with respect to* f.

Let $\mathcal{S}$ be an admissible set for f. Set $\mathcal{T}_{\mathcal{S}} = \mathcal{T} - (\mathcal{T} \cap \mathcal{S})$ to ease notation. Consider the following set of tuples of pairs:

$$\mathcal{X} = \left\{ (p, K_p)_{p \in \mathcal{W}} : \; K_p = K^{(2)}_{0,p}(p) \text{ or } K_p = K^{(2)}_p[p] \right\}.$$

Note $\#\mathcal{X} = 2 \cdot \#\mathcal{W}$. Given $x \in \mathcal{X}$, set

$$\mathcal{W}^{\mathrm{C}}_x = \left\{ p \in \mathcal{W} : (p, K_p) \in x,\, K_p = K^{(2)}_{0,p}(p) \right\}$$
$$\mathcal{W}^{\mathrm{Pm}}_x = \left\{ p \in \mathcal{W} : (p, K_p) \in x,\, K_p = K^{(2)}_p[p] \right\}.$$

Theorem 3.4 *Let m be a square-free positive integer, $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(m))$ a newform and $\mathcal{S}$ an admissible set for f. For each $x \in \mathcal{X}$, there is a cusp form $F^{\mathcal{S},x}_f \in S_\kappa(\Gamma^{\mathcal{S},x})$, unique up to scalar multiples, satisfying*

$$L(s, F^{\mathcal{S},x}_f, \text{spin}) = \left(\prod_{p \in \mathcal{S}} (1 - p^{-s+\kappa-1}) \right) \zeta(s-\kappa+1)\zeta(s-\kappa+2)L(s, f),$$

where $\Gamma^{\mathcal{S},x} = \mathrm{GSp}_4(\mathbb{Q}) \cap \mathrm{GSp}^+_4(\mathbb{R})K^{\mathcal{S},x}$ and

$$K^{\mathcal{S},x} = \prod_{\substack{p \in \mathcal{S} \cup \mathcal{W}^{\mathrm{C}}_x \\ p \nmid \infty}} K^{(2)}_{0,p}(p) \prod_{p \in \mathcal{T}_{\mathcal{S}} \cup \mathcal{W}^{\mathrm{Pm}}_x} K^{(2)}_p[p] \prod_{p \nmid m} \mathrm{GSp}_4(\mathbb{Z}_p).$$

We have $\eta_p F^{\mathcal{S},x}_f = -F^{\mathcal{S},x}_f$ for each $p \in \mathcal{T}_{\mathcal{S}}$ and $\eta_p F^{\mathcal{S},x}_f = F^{\mathcal{S},x}_f$ otherwise.

Proof The proof here follows from the arguments given in the proof of [13, Theorem 5.2]. We include part of it here for the convenience of the reader. Note throughout this proof when we write unique we mean unique up to scalar multiples.

Theorem 3.1 along with the fact that we are assuming $\mathcal{S}$ is admissible gives a representation $\Pi_f := \Pi(\pi_f \otimes \pi_S) = \otimes \Pi(\pi_{f,p} \otimes \pi_{S,p})$ whose local components are given by

(1) for $p = \infty$, $\Pi(\pi_{f,\infty} \otimes \pi_{\mathcal{S},\infty}) = \Pi(D(2\kappa-2) \otimes D(1))$;
(2) for $p \nmid m$, $\Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p})$ is an Iwahori-spherical representation $\Pi(\sigma, \chi_1, \chi_2)$ with σ, χ_1, and χ_2 unramified characters of $\mathbb{Q}_p^\times$;
(3)

$$\Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p}) = \begin{cases} \Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)}) & \text{for } p \in \mathcal{S} - \{\infty\}, \\ \Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}_{\mathrm{GL}(2)}) & \text{for } p \in \mathcal{T}_{\mathcal{S}}, \\ \Pi(\xi\, \mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}_{\mathrm{GL}(2)}) & \text{for } p \in \mathcal{W}. \end{cases}$$

It is well known that we can pick a unique lowest weight vector in $\Pi(\pi_{f,\infty} \otimes \pi_{\mathcal{S},\infty})$ and a unique $\mathrm{GSp}_4(\mathbb{Z}_p)$-fixed vector in $\Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p})$ for $p \nmid m$. For p a finite prime in $\mathcal{S}$, we have a unique vector in $\Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p})$ fixed by $K_{0,p}^{(2)}(p)$ by [13, Table 30]. Similarly, in the case $p \in \mathcal{T}_{\mathcal{S}}$, we have a unique vector fixed by $K_p^{(2)}[p]$. The last case to consider is for $p \in \mathcal{W}$. In this case there are two choices for vectors. One has a choice of a vector fixed by $K_{0,p}^{(2)}(p)$ or one fixed by $K_p^{(2)}[p]$. Upon making this choice, we have a unique fixed vector in $\Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p})$ of the correct level for each place so we can extract a Siegel cusp form $F_f^{\mathcal{S},x} \in S_\kappa(\Gamma^{\mathcal{S},x})$ as claimed.

The proof of the uniqueness (up to scalar factors) of $F_f^{\mathcal{S},x}$ in $S_\kappa(\Gamma^{\mathcal{S},x})$ goes exactly as in as in the proof of Theorem 5.2 in [13]. We include it for the convenience of the reader. Assume F' is another cusp form with the L-function as in the statement of the theorem. Let Φ' be the corresponding adelic function, generating a multiple of an automorphic representation Π'. From the form of Euler factors at good primes, we see that the local components of Π_f and Π' coincide almost everywhere. Using a theorem of Piatetski-Shapiro ([10, Theorem 2.2]), the representation Π' is also a lift of the form $\Pi(\pi' \otimes \pi_{S'})$ for some automorphic representation π' of $\mathrm{GL}_2(\mathbb{A})$ and some set of places S'. The local components of Π', being Iwahori-spherical, must therefore be among the ones occurring in [13, Table 30]. Looking at the Euler factors ([13, Table 33]), it shows that $\Pi(\pi_{f,\infty} \otimes \pi_{\mathcal{S},\infty}) = \Pi(\pi'_\infty \otimes \pi_{\mathcal{S}',\infty})$ and that, for $p \mid m$, $\Pi(\pi_{f,p} \otimes \pi_{\mathcal{S},p}) = \Pi(\pi'_p \otimes \pi_{\mathcal{S}',p})$. Therefore the global representations are isomorphic. Using the multiplicity one result ([10, Theorem 6.2]), the representations Π_f and Π' coincide as spaces of automorphic forms. Hence Φ_f and Φ' are elements of the same irreducible space of automorphic forms. The uniqueness now follows from the local uniqueness of vectors expressed by the one dimensionality of the spaces of fixed vectors [13, Table 30]. The statement about the Atkin–Lehner eigenvalues follows immediately from [13, Table 30].

The result on the L-functions now follows from the local L-functions defined above, switching from the automorphic normalizations of the L-functions to the modular normalizations. □

To see Theorem 3.1 recovers the congruence-level Saito–Kurokawa lift $F_f \in S_\kappa(\Gamma_0^{(2)}(m))$, we set $\mathcal{S} = \mathcal{T} \cup \{\infty\}$ and $x = \left((p, K_{0,p}^{(2)}(p))_{p\in\mathcal{W}}\right)$. To recover the paramodular Saito–Kurokawa lift, set $\mathcal{S} = \{\infty\}$ and $x = \left((p, K_p^{(2)}[p])_{p\in\mathcal{W}}\right)$.

4 Mixed level representation theoretic Saito–Kurokawa lifts

In this section we consider a special case of Theorem 3.4. In particular, we recover a classical case claimed in [8]. Keeping in mind the issues with [8] noted in the introduction, this section provides the first complete construction of such Saito–Kurokawa lifts.

Write $m = Mt$ with $\gcd(M, t) = 1$. Define

$$K_M^{(2)}[t] = \prod_{p|M} K_{0,p}^{(2)}(p) \prod_{p|t} K_p^{(2)}[p] \prod_{p\nmid Mt} \mathrm{GSp}_4(\mathbb{Z}_p).$$

Observe that we have $K^{\mathcal{S},x} = K_M^{(2)}[t]$ for $S = \{p \mid M : \epsilon_p = -1\} \cup \{\infty\}$ and x is chosen so that $\mathcal{W}_x^{\mathrm{C}} = \{p \in \mathcal{W} : p \mid M\}$ and $\mathcal{W}_x^{\mathrm{Pm}} = \{p \in \mathcal{W} : p \mid t\}$. Define

$$\Gamma_M[t] = \mathrm{GSp}_4(\mathbb{Q}) \cap \mathrm{GSp}_4^+(\mathbb{R}) K_M^{(2)}[t].$$

One can check that

$$\Gamma_M[t] = \mathrm{Sp}_4(\mathbb{Q}) \bigcap \left\{ \begin{pmatrix} \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t^{-1}\mathbb{Z} \\ M\mathbb{Z} & Mt\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ Mt\mathbb{Z} & Mt\mathbb{Z} & t\mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}.$$

As in [8] we consider the following subspace of cusp forms. (One should note that there is an error in the definition of the space $S_{2\kappa-2}^t(\Gamma_0(Mt))$ given in [8] where the Atkin–Lehner eigenvalue is given to be $(-1)^{\kappa-1}$ instead of $(-1)^{\kappa}$. With the definition given there, one does not recover the correct spaces upon setting $M = 1$.)

Definition 4.1 Let $t, M \in \mathbb{N}$ be square-free such that $\gcd(M, t) = 1$. We define the following subspace of elliptic cusp forms:

$$S_{2\kappa-2}^t(\Gamma_0(Mt)) = \left\{ f \in S_{2\kappa-2}(\Gamma_0(Mt)) : \epsilon_t = (-1)^{\kappa} \right\}.$$

Note that in case $M = 1$, $S_{2\kappa-2}^t(\Gamma_0(Mt)) = S_{2\kappa-2}^-(\Gamma_0(t))$, and in case $t = 1$, $S_{2\kappa-2}^t(\Gamma_0(Mt)) = S_{2\kappa-2}(\Gamma_0(M))$. In order to recover the lifting claimed in [8], we just need to show that the set $S = \{p \mid M : \epsilon_p = -1\} \cup \{\infty\}$ is admissible for any $f \in S_{2\kappa-2}^{t,\mathrm{new}}(\Gamma_0(Mt))$.

Corollary 4.2 *Let M and t be square-free integers,* $\gcd(M, t) = 1$*, and* $f \in S_{2\kappa-2}^{t,\mathrm{new}}(\Gamma_0(Mt))$ *a newform. There exists an eigenform $F_f \in S_\kappa(\Gamma_M[t])$, unique up to scalar multiples, satisfying*

$$L(s, F_f, \mathrm{spin}) = \left(\prod_{\substack{p|M \\ \epsilon_p=-1}} (1 - p^{-s+\kappa-1}) \right) \zeta(s-\kappa+1)\zeta(s-\kappa+2)L(s, f). \quad (1)$$

Moreover, for each $p \mid t$ we have $\eta_p F_f = \epsilon_p F_f$ and for each $p \mid M$ we have $\eta_p F_f = F_f$.

Proof It only remains to show that $\epsilon(1/2, \pi_f) = (-1)^{\#S}$. We have

$$\begin{aligned}\epsilon(1/2, \pi_f) &= i^{2\kappa-2}\epsilon_{tM}\\ &= (-1)^{\kappa-1}\epsilon_t\epsilon_M\\ &= (-1)^{\kappa-1}(-1)^{\kappa}\epsilon_M\\ &= (-1)(-1)^{\#\mathcal{S}-1}\\ &= (-1)^{\#\mathcal{S}}.\end{aligned}$$

□

Using SAGE [16] it is elementary to calculate examples of newforms in the space $S_{2\kappa-2}^{t,\text{new}}(\Gamma_0(Mt))$ for various κ, M, and t. For example, we have $f(z) = q - 256q^2 - 6561q^3 + 65536q^4 + 645150q^5 + O(q^6)$ and $g(z) = q - 256q^2 + 6561q^3 + 65536q^4 - 72186q^5 + O(q^6)$ both lying in the space $S_{18}^{2,\text{new}}(\Gamma_0(6))$. The above theorem then guarantees that we have forms F_f and F_g in $S_\kappa(\Gamma_M[t])$ satisfying Eq. (1).

References

1. Agarwal, M., Brown, J.: Saito–Kurokawa lifts of square-free level and lower bounds on Selmer groups. Math. Z. **276**(3), 889–924 (2014)
2. Agarwal, M., Brown, J.: Saito–Kurokawa lifts of odd square-free level. Kyoto J. Math. 1–24 (2014)
3. Asgari, M., Schmidt, R.: Siegel modular forms and representations. Manuscr. Math. **104**, 172–200 (2001)
4. Bump, D.: Automorphic Forms and Representations. Volume 55 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1998)
5. Gritsenko, V.: Irrationality of the moduli spaces of polarized abelian surfaces. In: Barth, W. (ed.) Abelian Varieties. Proceedings of the international conference held in Egloffstein, Germany, October 3–8, 1993, pp. 63–81. Walter de Gruyter, Berlin (1995)
6. Ibukiyama, T.: Saito-Kurokawa liftings of level N and practical construction of Jacobi forms. Kyoto J. Math. **52**(1), 141–178 (2012)
7. Longo, M., Nicole, H.: The Saito–Kurokawa lifting and Darmon points. Math. Ann. **356**(2), 469–486 (2013)
8. Manickam, M., Ramakrishnan, B.: On Shimura, Shintani and Eichler–Zagier correspondences. Trans. Am. Math. Soc. **352**, 2601–2617 (2000)
9. Manickam, M., Ramakrishnan, B., Vasudevan, T.C.: On Saito–Kurokawa descent for congruence subgroups. Manuscr. Math. **81**, 161–182 (1993)
10. Piatetski-Shapiro, I.I.: On the Saito–Kurokawa lifting. Invent. Math. **71**, 309–338 (1983)
11. Sally Jr, P., Tadić, M.: Induced representations and classifications for GSp(2, F) and Sp(2, F). Mém. Soc. Math. France (N.S.) **52**, 75–133 (1993)
12. Schmidt, R.: The Saito–Kurokawa lifting and functoriality. Am. J. Math. **127**, 209–240 (2005)
13. Schmidt, R.: On classical Saito–Kurokawa liftings. J. Reine Angew. Math. **604**, 211–236 (2007)
14. Skinner, C., Urban, E.: Sur les déformations p-adiques de certaines représentations automorphes. J. Inst. Math. Jussieu **5**, 629–698 (2006)
15. Skoruppa, N.-P., Zagier, D.: Jacobi forms and a certain space of modular forms. Invent. Math. **94**, 113–146 (1988)
16. Stein, W.A. et al.: Sage Mathematics Software (Version 5.10). The Sage Development Team. http://www.sagemath.org (2013). Accessed 24 Sep 2014
17. Zagier, D.: Sur la conjecture de Saito-Kurokawa. Sé Delange-Pisot-Poitou 1979/80. Volume 12 of Progress in Mathematics, pp. 371–394. Birkhauser, Boston-Basel-Stuttgart (1980)
18. Zantout, D.: Cuspidal mixed level Maass–Gritsenko lifts. pp. 1–40 (2013)