# SAITO-KUROKAWA LIFTS OF SQUARE-FREE LEVEL

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Abstract. Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a Hecke eigenform with $\kappa \geq 2$ even and $M \geq 1$ and odd and square-free. In this paper we survey the construction of the Saito-Kurokawa lifting from the classical and representation theoretic point of view. We also provide some arithmetic results on the Fourier coefficients of Saito-Kurokawa liftings. We then calculate the norm of the Saito-Kurokawa lift.

## 1. Introduction

It is well-established that one can prove deep theorems in arithmetic by studying liftings of automorphic forms from a reductive algebraic group to a larger reductive algebraic group. For instance, one can see Ribet's proof of the converse of Herbrand's theorem ([21]), Wiles' proof of the main conjecture of Iwasawa theory for totally real fields ([30]), or Skinner-Urban's proof of the main conjecture of Iwasawa theory for GL(2) ([28]) for three prominent examples of this philosophy. One such lifting that has figured prominently in several such results is the Saito-Kurokawa lifting that lifts a form from GL(2) to GSp(4). One can see [1, 4, 16, 27] for examples of arithmetic applications of Saito-Kurokawa liftings. It is these liftings that this paper focuses on.

The Saito-Kurokawa lifting in the full level case was established via a series of papers culminating in the work of Zagier ([31]). The lifting was established from an automorphic point of view via the work of Piatetski-Shapiro ([20]) and Schmidt ([24, 25]). For the arithmetic applications referenced above, one needs a classical construction of the Saito-Kurokawa lifting of square-free level. This lifting was claimed in a series of papers ([19, 17, 18]). Unfortunately, there are many omitted proofs in these papers and the generalized Maass lifting used in these papers is known to be given incorrectly there. It was not until recently that a correct treatment of the generalized Maass lifting was given by Ibukiyama ([10]) which allows one to give a correct classical construction of the Saito-Kurokawa lifting of square-free level.

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In this paper we survey the representation theory construction as well as the classical construction of the Saito-Kurokawa lift of square-free level. The representation theory construction appears in §3. This construction shows that given a newform $f \in S_{2\kappa-2}(\Gamma_0(M))$ for $M$ square-free and $\kappa \geq 2$ even there is a cuspform $F_f \in S_\kappa\left(\Gamma_0^{(2)}(M)\right)$ that is uniquely determined up to scalar multiples whose spinor $L$-function factors in terms of the Riemann zeta function and the $L$-function of $f$. One can see Theorem 3.3 for a precise statement. The classical results are put together in §4. In this section we also show that with a suitable choice of scalar one can fix a lifting $F_f$ that has Fourier coefficients in the same ring as $f$'s Fourier coefficients. This is essential for arithmetic applications. Finally, in §5 we compute the norm of $F_f$. Such a calculation originally appeared in [3], but this was based on the incorrect Maass lifting mentioned above so is not correct. This norm is needed for the main result of [1].

## 2. Definitions and notation

In this section we fix basic definitions and notations we will use throughout the rest of the paper. Given a ring $R$, we let $\mathrm{Mat}_n(R)$ denote the set of $n$ by $n$ matrices with entries in $R$. As usual we let $\mathrm{GL}(n,R) \subset \mathrm{Mat}_n(R)$ denote the group of invertible matrices and $\mathrm{SL}(n,R) \subset \mathrm{GL}(n,R)$ the matrices with determinant 1. We write $1_n$ for the identity matrix in $\mathrm{GL}_n(R)$ and $0_n$ for the zero matrix in $\mathrm{Mat}_n(R)$. Given $A \in \mathrm{Mat}_n(R)$, we denote the transpose of $A$ by ${}^tA$. Let $J = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$. The symplectic group $\mathrm{GSp}(4,R)$ is defined by

$$\mathrm{GSp}(4,R) = \{g \in \mathrm{GL}(4,R) : {}^tgJg = \mu(g)J, \mu(g) \in \mathrm{GL}(1,R)\}.$$

We set $\mathrm{Sp}(4,R) = \ker(\mu)$. We let $\mathrm{PGSp}(4,R)$ denote the projective symplectic group. For $M \geq 1$ an integer we let $\Gamma_0(M) \subset \mathrm{SL}(2,\mathbb{Z})$ have its usual meaning and set

$$\Gamma_0^{(2)}(M) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) : C \equiv 0 \pmod{M} \right\}.$$

We write $e(z) = e^{2\pi i z}$. We let $\mathfrak{h}^n = \{Z = X + iY \in \mathrm{Mat}_n(\mathbb{C}) : X, Y \in \mathrm{Mat}_n(\mathbb{R}), Y > 0\}$. Let $\kappa \geq 2$ be an integer and $M \geq 1$ a square-free integer. We let $S_\kappa(\Gamma_0(M))$ denote the cusp forms of weight $\kappa$ and level $\Gamma_0(M)$. Let $f \in S_\kappa(\Gamma_0(M))$ be a normalized eigenform with Fourier expansion

$$f(z) = \sum_{n \geq 1} a_f(n) e(nz).$$

Given a ring $R$, we write $S_\kappa(\Gamma_0(M); R)$ to denote the space of cusp forms that have Fourier coefficients in $R$. We define the Peterson product on $S_\kappa(\Gamma_0(M))$ by setting

$$\langle f_1, f_2 \rangle = \frac{1}{[\mathrm{SL}(2,\mathbf{Z}) : \Gamma_0(M)]} \int_{\Gamma_0(M)\backslash\mathfrak{h}} f_1(z)\overline{f_2(z)} y^{\kappa-2} dx dy$$

for $f_1, f_2 \in S_\kappa(\Gamma_0(M))$.

We denote the cuspidal automorphic representation associated to $f$ by $\pi_f = \otimes' \pi_{f,p}$. Recall that $\pi_{f,\infty}$ is the discrete series representation with lowest weight vector of weight $\kappa$ and for $p \nmid M$ we have $\pi_{f,p}$ is the unramified principal series representation. The local representations for $p \mid M$ are determined by the Atkin-Lehner eigenvalues of $f$. For $p \mid M$, recall the Atkin-Lehner operator at $p$ is the matrix $W_p = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. If $f \in S_\kappa^{\text{new}}(\Gamma_0(M))$, we let $\epsilon_p \in \{\pm 1\}$ denote the Atkin-Lehner eigenvalue of $f$ at $p$, i.e., $W_p f = \epsilon_p f$. If $\epsilon_p = -1$, then $\pi_{f,p} = \mathrm{St}_{\mathrm{GL}(2)}$ and if $\epsilon_p = 1$ then $\pi_{f,p} = \xi\,\mathrm{St}_{\mathrm{GL}(2)}$ where $\mathrm{St}_{\mathrm{GL}(2)}$ is the Steinberg representation and $\xi$ is the unique non-trivial unramified quadratic character of $\mathbb{Q}_p^\times$.

We will also need $L$-functions attached to $f$ and $\pi_f$. For each prime $p \nmid M$ there exists a character $\sigma_p$ so that $\pi_{f,p} = \pi(\sigma_p, \sigma_p^{-1})$ (see [5, Section 4.5].) The $p$-Satake parameter of $f$ is given by $\alpha_0(p; f) = \sigma_p(p)$. We have

$$L(s, \pi_{f,p}) = (1 - \alpha_0(p; f)p^{-s})^{-1}(1 - \alpha_0(p; f)^{-1}p^{-s})^{-1}.$$

For $p \mid M$ we have

$$L(s, \pi_{f,p}) = (1 + \epsilon_p p^{-s-\frac{1}{2}})^{-1}.$$

We set

$$L_\infty(s, \pi_{f,\infty}) = (2\pi)^{-(s+(\kappa-1)/2)}\Gamma\left(s + (\kappa - 1)/2\right).$$

Set

$$L(s, \pi_f) = \prod_p L(s, \pi_{f,p}).$$

The functional equation for $L(s, \pi_f)$ is given by

$$L(s, \pi_f) = \varepsilon(s, \pi_f)L(1 - s, \pi_f)$$

where $\epsilon(s, \pi_f) = \prod_p \epsilon_p(s, \pi_{f,p})$ and

$$\varepsilon_p(s, \pi_{f,p}) = \begin{cases} (-1)^{\kappa/2} & \text{if } p = \infty, \\ -p^{\frac{1}{2}-s} & \text{if } \epsilon_p = -1, p < \infty, \\ p^{\frac{1}{2}-s} & \text{if } \epsilon_p = 1, p < \infty. \end{cases}$$

In particular, the sign of the functional equation is given by $\varepsilon(\frac{1}{2}, \pi_f) \in \{\pm 1\}$. The classical $L$-function of $f$ is given by

$$L(s, f) = \prod_{p<\infty} L\left(s + \frac{1}{2} - \kappa/2, \pi_{f,p}\right).$$

The completed $L$-function is denoted by

$$\Lambda(s, f) = L\left(s + \frac{1}{2} - \kappa/2, \pi_f\right).$$

Let $S_\kappa(\Gamma_0^{(2)}(M))$ denote the space of Siegel modular forms of weight $\kappa$ and level $\Gamma_0^{(2)}(M)$. Given $F \in S_\kappa(\Gamma_0^{(2)}(M))$, we have a Fourier expansion

$$F(z) = \sum_{T \in \Lambda_2} a_F(T) e(\mathrm{Tr}(Tz))$$

where $\Lambda_2$ is the set of 2 by 2 half integral positive definite symmetric matrices. As above, for a ring $R$ we write $S_\kappa(\Gamma_0^{(2)}(N); R)$ to denote the forms with Fourier coefficients in $R$.

Given $F \in S_\kappa(\Gamma_0^{(2)}(M))$, there is a cuspidal automorphic representation $\Pi_F$ of $\mathrm{PGSp}(4, \mathbb{A})$ associated to $F$. We can decompose it into local components $\Pi_F = \otimes \Pi_{F,p}$ with $\Pi_{F,p}$ a representation of $\mathrm{PGSp}(4, \mathbb{Q}_p)$. We refer the reader to [2, Section 3] for the details concerning the construction of cuspidal automorphic representations associated to Siegel cusp forms. For all but finitely many places $p$ the representation $\Pi_{F,p}$ will be an Iwahori spherical representation $\Pi(\sigma, \chi_1, \chi_2)$, which is isomorphic to the Langlands quotient of an induced representation of the form $\chi_1 \times \chi_2 \rtimes \sigma$ with $\chi_i$ and $\sigma$ unramified characters of $\mathbb{Q}_p^\times$. One can see [2, 22] for the definitions and details. For such $p$ the $p$-Satake parameters are defined by $b_0 = \sigma(p)$ and $b_i = \chi_i(p)$ for $i = 1, 2$. We define

$$L(s, \Pi_{F,p}, \mathrm{spin}) = \big((1 - b_0 p^{-s})(1 - b_0 b_1 p^{-s})(1 - b_0 b_2 p^{-s})(1 - b_0 b_1 b_2 p^{-s})\big)^{-1}$$

for $\Pi_{F,p} = \Pi(\sigma, \chi_1, \chi_2)$. We leave the local $L$-functions for $p = \infty$ and $p \mid M$ undefined for now as these will be given in Section 3. Set

$$L(s, \Pi_F, \mathrm{spin}) = \prod_p L(s, \Pi_{F,p}, \mathrm{spin}).$$

The classical spinor $L$-function is given by

$$L(s, F, \mathrm{spin}) = \prod_{p<\infty} L(s - \kappa + 3/2, \Pi_{F,p}, \mathrm{spin}).$$

The completed $L$-function is denoted by

$$\Lambda(s, F, \mathrm{spin}) = L(s - \kappa + 3/2, \Pi_F, \mathrm{spin}).$$

## 3. Representation theoretic construction

In this section we recall the representation theoretic construction by Schmidt [25] of the Saito-Kurokawa lift in the square free congruence level case. Let $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ be a newform of weight $2\kappa - 2$ where $\kappa \geq 2$ is even and $M$ is square free. Assume that $f$ is an eigenform for all Hecke operators $T(p)$ with $p \nmid M$. Let $\pi_f$ be the cuspidal automorphic representation of $\mathrm{PGL}(2, \mathbb{A})$ attached to $f$. Let $\Sigma$ be the set of primes $p$ dividing $M$ including $\infty$ and let $S \subset \Sigma$ such that $\infty \in S$. Since $M$ is square free, the local components $\pi_{f,p}$ of $\pi_f$ are square-integrable if and only if $p|M$ or $p = \infty$.

Define $\pi_S = \otimes' \pi_{S,p}$ in $\mathrm{PGL}(2, \mathbb{A})$ by

$$\pi_{S,p} = \begin{cases} \mathbf{1}_{\mathrm{GL}(2)} & \text{if } p \notin S \\ \mathrm{St}_{\mathrm{GL}(2)} & \text{if } p \in S \end{cases}$$

where $\mathbf{1}_{\mathrm{GL}(2)}$ is the trivial representation and $\mathrm{St}_{\mathrm{GL}(2)}$ is the Steinberg representation in $\mathrm{PGL}(2)$. Then $\pi_S$ is an automorphic representation being an irreducible constituent of a globally induced representation.

**Theorem 3.1.** [25, Theorem 1.1] *Let $\pi_f = \otimes' \pi_{f,p}$ be the cuspidal automorphic representation of $\mathrm{PGL}(2, \mathbb{A})$ corresponding to a newform $f \in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(M))$ where $\kappa \geq 2$ is even and $M$ is square-free. Let $S$ be a subset of places of $\mathbb{Q}$ containing $\infty$ and dividing $M$ and define $\pi_S$ as above. Let $\varepsilon(s, \pi_f)$ be the Euler factor associated to $\pi_f$. If*

$$(-1)^{\#S} = \varepsilon\left(\frac{1}{2}, \pi_f\right)$$

*then the global lifting $\Pi(\pi_f \otimes \pi_S)$ is a cuspidal automorphic representation of $\mathrm{PGSp}(4, \mathbb{A})$ which appears discretely in the space of automorphic forms.*

One can think of $\Pi(\pi_f \otimes \pi_S)$ as a functorial lifting of the representation $\pi_f \otimes \pi_S$ on $\mathrm{PGL}(2, \mathbb{A}) \times \mathrm{PGL}(2, \mathbb{A})$ to $\mathrm{PGSp}(4, \mathbb{A})$. By Theorem 3.1, after a suitable choice of $S$ (to be made precise later) that satisfies the hypothesis we obtain a cuspidal automorphic representation $\Pi = \Pi(\pi_f \otimes \pi_S)$ in $\mathrm{PGSp}(4, \mathbb{A})$. Let $\Pi = \otimes'_{p \leq \infty} \Pi_p$ be the factorization of $\Pi$ into local representations. Next we study the local representations $\Pi_p$ with the aim of controlling the weight and the level of the lift $\Pi$.

Since $f$ has weight $2\kappa - 2$, the archimedean component $\pi_{f,\infty}$ of $\pi_f$ is $\mathcal{D}(2\kappa - 3)$, the holomorphic discrete series representation of $\mathrm{PGL}(2, \mathbb{R})$ with a lowest weight vector of weight $2\kappa - 2$. The archimedean component of $\mathrm{St}_{\mathrm{GL}(2)}$ is $\mathrm{St}_\infty := \mathcal{D}(1)$ the lowest discrete series representation of $\mathrm{PGL}(2)$, with lowest weight vector of weight 2. Hence

$$\Pi_\infty = \Pi(\pi_{f,\infty} \otimes \mathrm{St}_{\mathrm{GL}(2)}) = \Pi(\mathcal{D}(2\kappa - 3) \otimes \mathcal{D}(1))$$

and it is the holomorphic discrete series representation of $\mathrm{PGSp}(4, \mathbb{R})$ with scalar minimal $K$-type $(\kappa, \kappa)$ for $\kappa > 2$ (see §4,[24]). Let $\Phi_\infty$ be the distinguished lowest weight vector in it.

At the finite places, let $\Phi_p \in \Pi_p$, be vectors chosen such that they are fixed by compact open subgroups $K_p \subset \mathrm{GSp}(4, \mathbb{Q}_p)$. Let $\Phi = \otimes'_{p \leq \infty} \Phi_p \in \Pi$. Then $\Phi$ corresponds to a classical homomorphic cusp form $F_f$ of weight $\kappa$ with respect to the congruence subgroup $\Gamma = \mathrm{GSp}(4, \mathbb{Q}) \cap \mathrm{GSp}^+(4, \mathbb{R}) \prod_{p < \infty} K_p \subset \mathrm{Sp}(4, \mathbb{Q})$.

Having found the weight, we will now determine the level of $F_f$. If $p \nmid M$ then the local representation $\Pi_p$ is $\Pi(\pi_{f,p} \otimes \mathbf{1}_{\mathrm{GL}(2)})$, where $\pi_{f,p}$ is the unramified principal series representation containing non-zero vectors fixed by $\mathrm{GL}(2, \mathbb{Z}_p)$. Hence $\Pi_p$ is an unramified representation and we can choose $K_p$ as $K_p = \mathrm{GSp}(4, \mathbb{Z}_p)$ and $\Phi_p$ as a spherical vector. For primes $p$ dividing

$M$ where $M$ is square free, the local components $\pi_{f,p}$ of $\pi_f$ for $p|M$ can only be the Steinberg representation $\mathrm{St}_{\mathrm{GL}(2),p}$ or a non-trivial unramified twist $\xi\,\mathrm{St}_{\mathrm{GL}(2),p}$ where $\xi$ is the unique non-trivial unramified quadratic character of $\mathbb{Q}_p^\times$. Hence one is led to study the following local representations:

(1) $\Pi(\mathrm{St}_{\mathrm{GL}(2),p} \otimes \mathbf{1}_{\mathrm{GL}(2),p})$
(2) $\Pi(\mathrm{St}_{\mathrm{GL}(2),p} \otimes \mathrm{St}_{\mathrm{GL}(2),p})$
(3) $\Pi(\xi\,\mathrm{St}_{\mathrm{GL}(2),p} \otimes \mathbf{1}_{\mathrm{GL}(2),p})$
(4) $\Pi(\xi\,\mathrm{St}_{\mathrm{GL}(2),p} \otimes \mathrm{St}_{\mathrm{GL}(2),p})$

We can rule out $\Pi(\xi\,\mathrm{St}_{\mathrm{GL}(2),p} \otimes \mathrm{St}_{\mathrm{GL}(2),p})$ as a possible local representation at a place dividing $M$ since it is supercuspidal by paper [11]. A supercuspidal local representation cannot occur as a local component of an automorphic representation associated to a Siegel modular since it has no Iwahori fixed vectors.

As we are only interested in congruence level Saito-Kurokawa lifts, we now make precise our choice of $S$. Let

$$S = \left\{\infty\} \cup \{p|M : \varepsilon\left(\frac{1}{2}, \pi_{f,p}\right) = -1\right\}.$$

We note that this choice of $S$ satisfies the hypotheses of Theorem 3.1:

$$\varepsilon\left(\frac{1}{2}, \pi_f\right) := (-1)^{\kappa-1} \prod_{p|M} \varepsilon\left(\frac{1}{2}, \pi_{f,p}\right) = (-1)^{\kappa-1} \prod_{p \in S} -1 = (-1)^{\#S-\kappa-2} = (-1)^{\#S}$$

since $\kappa$ is even.

**Remark 3.2.** For $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$, $\pi_{f,p}$ is a representation of $\mathrm{PGL}(2, \mathbb{Q}_p)$. Since representations of $\mathrm{PGL}(2, \mathbb{Q}_p)$ are self-dual, the $\varepsilon$-factor at the central point $\frac{1}{2}$ is $\pm 1$ (see [23, Lemma 3.2.1]). At the same time, the Atkin-Lehner involution acts on the one-dimensional space spanned by the newform. This involution also defines a sign. In [23], Schmidt shows that the two signs attached to $\pi_{f,p}$ are the same, i.e., $\epsilon_p = \varepsilon(\frac{1}{2}, \pi_{f,p})$. Hence we have characterized $S$ in terms of Euler factors instead of Atkin-Lehner eigenvalues as done in [25].

Since $\varepsilon(\frac{1}{2}, \pi_1 \otimes \pi_2) = \varepsilon(\frac{1}{2}, \pi_1)\varepsilon(\frac{1}{2}, \pi_2)$, $\varepsilon(\frac{1}{2}, \mathbf{1}_{\mathrm{GL}(2),p}) = 1$, $\varepsilon(\frac{1}{2}, \mathrm{St}_{\mathrm{GL}(2),p}) = -1$, and $\varepsilon(\frac{1}{2}, \xi\,\mathrm{St}_{\mathrm{GL}(2),p}) = 1$ by our choice of $S$ the only possible local representations are $\Pi(\mathrm{St}_{\mathrm{GL}(2),p} \otimes \mathrm{St}_{\mathrm{GL}(2),p})$ and $\Pi(\xi\,\mathrm{St}_{\mathrm{GL}(2),p} \otimes \mathbf{1}_{\mathrm{GL}(2),p})$. Hence we can limit our analysis to these cases.

A detailed study of the invariant vectors in $\Pi(\mathrm{St}_{\mathrm{GL}(2),p} \otimes \mathrm{St}_{\mathrm{GL}(2),p})$ and $\Pi(\xi\,\mathrm{St}_{\mathrm{GL}(2),p} \otimes \mathbf{1}_{\mathrm{GL}(2),p})$ is carried out in §2 of [25] with the results summarized in Table (30) of [25]. The table shows that there is a unique local fixed vector for the Siegel parabolic of level $p$ in each of the above cases. Since this is the case for all $p|M$ and $M$ is square free, the Siegel form $F_f$ has level $\Gamma_0^{(2)}(M)$ and being cuspidal of scalar weight $\kappa$ it is in $S_\kappa(\Gamma_0^{(2)}(M))$. Finally, it can be shown that such a $F_f$ is unique up to scalar multiples.

We have the local $L$-factors associated to $\pi_S$ are given by

$$L(s,\pi_{S,p}) = \begin{cases} (2\pi)^{-s-\frac{1}{2}}\Gamma\left(s+\frac{1}{2}\right) & p=\infty \\ (1-p^{-s-\frac{1}{2}})^{-1} & p\in S,\, p<\infty \\ (1-p^{-s-\frac{1}{2}})^{-1}(1-p^{-s+\frac{1}{2}})^{-1} & p\notin S. \end{cases}$$

We also make use of the fact that

$$L(s,\mathbf{1}_{\mathrm{GL}(2),\infty}) = 2(2\pi)^{-s+\frac{1}{2}}\Gamma\left(s-\frac{1}{2}\right).$$

This allows us to conclude the global $L$-function of $\pi_S$ is given by

$$L(s,\pi_S) = Z\left(s+\frac{1}{2}\right)Z\left(s-\frac{1}{2}\right)\prod_{p\in S}\frac{L(s,\mathrm{St}_{\mathrm{GL}(2),p})}{L(s,\mathbf{1}_{\mathrm{GL}(2),p})}$$

where

$$\frac{L(s,\mathrm{St}_{\mathrm{GL}(2),p})}{L(s,\mathbf{1}_{\mathrm{GL}(2),p})} = \begin{cases} \frac{1}{4\pi}(s-\frac{1}{2}) & p=\infty \\ 1-p^{-s+\frac{1}{2}} & p<\infty \end{cases}$$

and $Z(s)$ is the completed Riemann zeta function. From this, along with the description of $L(s,\pi_f)$ given in the previous section and Theorem 3.1, we have the following theorem.

**Theorem 3.3.** *(*[25, Theorem 5.2]*) Let $\kappa\geq 2$ be even and $M$ a square-free positive integer. Let $f\in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(M))$ an elliptic newform. Then there exists a cusp form $F_f\in S_\kappa(\Gamma_0^{(2)}(M))$, unique up to scalar multiples, whose spinor L-function is given by*

$$L(s,\pi_{F_f},\mathrm{spin}) = \frac{1}{4\pi}\left(s-\frac{1}{2}\right)\left(\prod_{\varepsilon(\frac{1}{2},\pi_{f,p})=-1}(1-p^{-s+\frac{1}{2}})\right)Z\left(s+\frac{1}{2}\right)Z\left(s-\frac{1}{2}\right)L(s,\pi_f).$$

## 4. Classical construction

In the previous section we summarized the results of [25] showing the existence of a Saito-Kurokawa lift of square-free level from the automorphic forms viewpoint. In this section we gather known results and give a classical construction of the Saito-Kurokawa lifting from $S_{2\kappa-2}(\Gamma_0(M))$ to $S_\kappa(\Gamma_0^{(2)}(M))$ for $\kappa\geq 2$ an even integer and $M\geq 1$ an odd square-free integer. Theorem 3.3 gives a Saito-Kurokawa lifting that is unique up to scalars. Using this classical construction we fix the scalar so that the resulting Saito-Kurokawa lift is more useful for arithmetic applications (see for example [1, 16]). In particular, given a normalized Hecke eigenform $f\in S_{2\kappa-2}(\Gamma_0(M))$, if we let $\mathcal{O}$ be a ring containing the Hecke eigenvalues of $f$, we show the Saito-Kurokawa lift of can be normalized so that it has Fourier coefficients lying in $\mathcal{O}$ as well.

The classical lifting is constructed via a composition of liftings, the first from integral to half-integral weight, then from half-integral weight to Jacobi forms, and finally from Jacobi forms to Siegel modular forms. We begin by recalling results on the lifting from integral to half-integral weight forms.

Let $D < 0$ be a fundamental discriminant and let $\theta_{\kappa,D} : S_{2\kappa-2}(\Gamma_0(M)) \to S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ be Shintani's lifting ([26]). One has that $\theta_{\kappa,D}$ gives a Hecke-equivariant isomorphism between $S_{2\kappa-2}(\Gamma_0(M))$ and $S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ ([12, § 5, Theorem 2]).

Let $\mathcal{O}$ be a ring so that an embedding of $\mathcal{O}$ into $\mathbb{C}$ exists. We choose such an embedding and identify $\mathcal{O}$ with its image in $\mathbb{C}$. Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a normalized Hecke eigenform and assume $\mathcal{O}$ contains the Fourier coefficients of $f$. The Shintani lifting of $f$ is determined up to a scalar multiple. In [29] Stevens' constructs a cohomological version of the Shintani lifting as a step in producing a $\Lambda$-adic Shintani lifting. This cohomological Shintani lifting allows one to conclude the following result.

**Theorem 4.1.** *([29, Prop. 2.3.1]) Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a Hecke eigenform. Let $D < 0$ be a fundamental discriminant. If the Fourier coefficients of $f$ are in $\mathcal{O}$, there exists a Shintani lifting $\theta^{\mathrm{alg}}_{\kappa,D}(f)$ with Fourier coefficients in $\mathcal{O}$. Moreover, if $\mathcal{O}$ happens to be a discrete valuation ring, we can normalize the Shintani lifting to have Fourier coefficients in $\mathcal{O}$ with at least one Fourier coefficient in $\mathcal{O}^\times$.*

This gives what we need for the first part of the construction. We now consider the lifting from half-integral weight forms to Jacobi forms. We recall the bijection between $J^c_{\kappa,1}(\Gamma_0(M)^J)$ and $S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ for $M$ an odd integer.

Let $g \in S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$. Define $g_j(\tau)$ by

$$g_j(\tau) = \sum_{\substack{n>0 \\ n\equiv j \pmod 4}} a_g(n)e(n\tau/4)$$

for $j = 0, 1$ where the $a_g(n)$ are the Fourier coefficients of $g$. Note we have $g(\tau) = g_0(4\tau) + g_1(4\tau)$. Define

$$\vartheta_j(\tau, z) = \sum_{n\in\mathbb{Z}} e\left(\frac{2n - j^2}{4}\tau + (2n - j)z\right).$$

Define a map $\mathcal{J}$ by

$$\mathcal{J}(g)(\tau, z) = g_0(\tau)\vartheta_0(\tau, z) + g_1(\tau)\vartheta_1(\tau, z)$$

for $g \in S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$. We have the following theorem.

**Theorem 4.2.** *([15, Corollary 3]) The map $\mathcal{J}$ gives an isomorphism between $S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ and $J^c_{\kappa,1}(\Gamma_0(M)^J)$.*

One immediately obtains that if the Fourier coefficients of $g$ lie in a ring $\mathcal{O}$, so do the Fourier coefficients of $\mathcal{J}(g)$. Moreover, if $g$ is an eigenform, so is $\mathcal{J}(g)$ with the same eigenvalues.

Finally, we recall the Maass lifting from the Jacobi forms to Siegel forms. Given a positive integer $m$, we define

$$\Delta_{M,0}(m) = \left\{ \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bcM = m, \gcd(a, M) = 1 \right\}.$$

Let

$$V_m : J^c_{\kappa,t}(\Gamma_0(M)^J) \to J^c_{\kappa,mt}(\Gamma_0(M)^J)$$

be the index shifting operator defined by

$$(V_m\phi)(\tau, z) = m^{\kappa-1} \sum_{g \in \Gamma_0(M)\backslash \Delta_{M,0}(m)} (\phi|_{\kappa,t} g)(\tau, z).$$

If the Fourier expansion of $\phi \in J^c_{\kappa,1}(\Gamma_0(M)^J)$ is given by

$$\phi(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2 \pmod 4}} c(D, r) e\left(\frac{r^2 - D}{4}\tau + rz\right),$$

then

$$(V_m\phi)(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2 \pmod{4m}}} \left( \sum_{\substack{d \mid \gcd(r,m) \\ \gcd(d,M)=1 \\ D\equiv r^2 \pmod{4md}}} d^{\kappa-1} c\left(\frac{D}{d^2}, \frac{r}{d}\right) \right) e\left(\frac{r^2 - D}{4m}\tau + rz\right).$$

Define a function on $\mathfrak{h}^2$ by

$$(\mathcal{V}_M\phi)(Z) = \sum_{m=1}^{\infty} (V_m\phi)(\tau, z) e(m\tau')$$

where $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$. One has the following result of Ibukiyama.

**Theorem 4.3.** *(*[10, Theorems 3.2, 4.1]*) The map $\mathcal{V}_M$ is an injective linear map from $J^c_{\kappa,1}(\Gamma_0(M)^J)$ to $S_\kappa(\Gamma_0^{(2)}(M))$. Moreover one has for $p \nmid M$*

$$T_S(p)(\mathcal{V}_M\phi) = \mathcal{V}_M\left(T_J(p)\phi + (p^{\kappa-1} + p^{\kappa-2})\phi\right),$$
$$T_S'(p)(\mathcal{V}_M\phi) = \mathcal{V}_M\left((p^{\kappa-2} + p^{\kappa-1})T_J(p)\phi + (2p^{2\kappa-3} + p^{2\kappa-4})\phi\right)$$

*where $T_S'(p) = pT_S(p^2) + p(1 + p + p^2)T(\operatorname{diag}(p, p, p, p))$. If $p \nmid M$, we have*

$$U_S(p)(\mathcal{V}_M\phi) = \mathcal{V}_M(U_J(p)\phi).$$

An immediate consequence of the calculation of the Fourier coefficients of $\mathcal{V}_M$ carried out in [10] is the following corollary.

**Corollary 4.4.** *Let $\phi \in J^c_{\kappa,1}(\Gamma_0(M)^J, \mathcal{O})$ for some ring $\mathcal{O}$. Then $\mathcal{V}_M\phi \in S_\kappa(\Gamma_0^{(2)}(M), \mathcal{O})$.*

Combining the result on the Shintani lifting with Theorems 4.2 and 4.3 we have the following theorem giving the existence of a Saito-Kurokawa lifting.

**Theorem 4.5.** *Let $\kappa \geq 2$ be an even integer and $M \geq 1$ an odd square-free integer. Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a normalized Hecke eigenform. Then there exists a nonzero cuspidal Siegel eigenform $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$ satisfying*

$$L^M(s, F_f, \mathrm{spin}) = \zeta^M(s-\kappa+1)\zeta^M(s-\kappa+2)L^M(s,f).$$

*Moreover, if $\mathcal{O}$ is a ring that can be embedded into $\mathbb{C}$ and $f$ has Fourier coefficients in $\mathcal{O}$, the lift $F_f$ can be normalized to have Fourier coefficients in $\mathcal{O}$. If $\mathcal{O}$ is a DVR, $F_f$ can be normalized to have Fourier coefficients in $\mathcal{O}$ with at least one Fourier coefficient in $\mathcal{O}^\times$.*

One should note here that the factorization of the spinor $L$-function given in Theorem 4.5 matches that of Theorem 3.3 once one switches from the automorphic to the classical $L$-functions and removes the infinite place and those places dividing $M$.

## 5. Norm of $F_f$

We now calculate the norm of $F_f$ in terms of the norm of $f$. This forms a key step in the main result of [1], but is also of independent interest. We do this by relating the norm of the image of each lift to the norm of the form being lifted for each of the three lifts composed to give the Saito-Kurokawa lift. Again, we fix $\kappa \geq 2$ to be an even integer and $M \geq 1$ to be odd and square-free.

Let $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ be a newform. For each prime $\ell \mid M$, recall $W_\ell$ is the Atkin-Lehner involution on $S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$. Define $\epsilon_\ell \in \{\pm 1\}$ by

$$f \mid W_\ell = \epsilon_\ell f.$$

We have the following theorem relating the norm of $f$ to that of $\theta_{\kappa,D}^{\mathrm{alg}}(f)$.

**Theorem 5.1.** [13, Corollary 1] *Let $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ be a newform and let $D < 0$ be a fundamental discriminant. Suppose that for all primes $\ell \mid M$ we have $\left(\frac{D}{\ell}\right) = \epsilon_\ell$. Then*

$$\frac{\left|a_{\theta_{\kappa,D}^{\mathrm{alg}}(f)}(|D|)\right|^2}{\langle \theta_{\kappa,D}^{\mathrm{alg}}(f), \theta_{\kappa,D}^{\mathrm{alg}}(f)\rangle} = 2^{\nu(M)} \frac{(\kappa-2)!}{\pi^{\kappa-1}} |D|^{\kappa-3/2} \frac{L(\kappa-1, f, \chi_D)}{\langle f, f\rangle} \tag{1}$$

*where $\chi_D = \left(\frac{D}{\cdot}\right)$ and $\nu(M)$ is the number of prime divisors of $M$.*

One should note that if there is a prime $\ell \mid M$ so that $\left(\frac{D}{\ell}\right) \neq \epsilon_\ell$, then one has $a_{\theta_{\kappa,D}^{\mathrm{alg}}(f)}(|D|) = 0$.

Now let $g \in S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ and $\mathcal{J}(g)$ the associated form in $J^c_{\kappa,1}(\Gamma_0(M)^J)$. Let $g(z) = \sum_{n=1}^{\infty} a_g(n)e(nz)$ be the Fourier expansion of $g$. Consider the summation $\sum_{n=1}^{\infty} \frac{a_g(n)^2}{n^{s+\kappa-3/2}}$. Applying the Rankin-Selberg method to this summation we have for sufficiently large $s$:

$$\begin{aligned} \frac{\Gamma(s+\kappa-3/2)}{(4\pi)^{s+\kappa-\frac{1}{2}}} \sum_{n=1}^{\infty} \frac{a_g(n)^2}{n^{s+\kappa-3/2}} &= \int_{\mathfrak{h}^1/\Gamma_\infty} |g(z)|^2 y^{s+\kappa-5/2} dxdy \\ &= \int_{\mathfrak{h}^1/\Gamma_0(4M)} y^{\kappa-\frac{1}{2}} |g(z)|^2 E_s^{4M}(z) \frac{dxdy}{y^2} \end{aligned}$$

where $E_s^{4M}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4M)} (\mathrm{Im}\,(\gamma z))^s$ and $\Gamma_\infty$ the stabilizer of $\infty$. In other words,

$$\sum_{n=1}^{\infty} \frac{a_g(n)^2}{n^{s+\kappa-3/2}} = \frac{(4\pi)^{s+\kappa-\frac{1}{2}}}{\Gamma(s+\kappa-3/2)} \int_{\Gamma_0(4M)\backslash\mathfrak{h}^1} E_s^{4M}(z) g(z) \overline{g(z)} y^{\kappa-\frac{1}{2}} \frac{dxdy}{y^2}. \tag{2}$$

Taking residues at $s=1$ we obtain

$$\begin{aligned} \operatorname*{res}_{s=1} \left( \sum_{n=1}^{\infty} \frac{a_g(n)^2}{n^{s+\kappa-3/2}} \right) &= \frac{(4\pi)^{1+\kappa-\frac{1}{2}} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]}{\Gamma(\kappa-\frac{1}{2})} \langle g, g \rangle \operatorname*{res}_{s=1} E_s^{4M}(z) \\ &= \frac{3 \cdot 2^{\kappa-1} (4\pi)^{\kappa-\frac{1}{2}}}{\pi^{3/2}(2\kappa-3)!!} \langle g, g \rangle \end{aligned}$$

where

$$n!! = \begin{cases} n(n-2)\ldots 5\cdot 3\cdot 1 & n>0, \text{ odd} \\ n(n-2)\ldots 6\cdot 4\cdot 2 & n>0, \text{ even} \end{cases}$$

and we have used that

$$\begin{aligned} \operatorname*{res}_{s=1} E_s^{4M}(z) &= \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]} \operatorname*{res}_{s=1} E_s(z) \\ &= \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]} \left( \frac{3}{\pi} \right) \end{aligned}$$

where $E_s(z)$ is the Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$. Solving the above residue calculation for $\langle g, g \rangle$ we obtain

$$\langle g, g \rangle = \frac{(2\kappa-3)!!}{3 \cdot 2^{3\kappa-2} \pi^{\kappa-2}} \operatorname*{res}_{s=1} \left( \sum_{n=1}^{\infty} \frac{a_g(n)^2}{n^{s+\kappa-3/2}} \right). \tag{3}$$

Recall the two half-integral weight modular forms $g_0$ and $g_1$ defined in § 4. Applying the same process to $g_0$ and $g_1$ we obtain

$$\langle g_j, g_j \rangle = \frac{(2\kappa-3)!!}{3 \cdot 2^{3\kappa-2}\pi^{\kappa-2}} \cdot 2^{2\kappa-1} \operatorname*{res}_{s=1} \left( \sum_{n \equiv j} \frac{a_g(n)^2}{n^{s+\kappa-3/2}} \right).$$

Thus we have

$$\langle g_0, g_0 \rangle + \langle g_1, g_1 \rangle = 2^{2\kappa-1} \langle g, g \rangle. \tag{4}$$

We need a slight generalization of Theorem 5.3 in [8]. In [8], the formula given only deals with the case $M = 1$. However, the proof carries through verbatim to the general case.

**Theorem 5.2.** *([8], Theorem 5.3) For $\mathcal{J}(g)$ and $g_j$ as defined above, one has*

(5)

$$\langle \mathcal{J}(g), \mathcal{J}(g) \rangle = \frac{1}{2\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)]} \int_{\Gamma_0(M)\backslash \mathfrak{h}^1} \sum_{j=0}^{1} g_j(z)\overline{g_j(z)} v^{\kappa-3/2} \frac{du dv}{v^2}.$$

Combining Equations 4 and 5 we have:

**Lemma 5.3.** *For $\mathcal{J}(g)$ and $g$ defined as above we have*

$$\langle \mathcal{J}(g), \mathcal{J}(g) \rangle = \frac{2^{2\kappa-2}}{[\Gamma_0(M) : \Gamma_0(4M)]} \langle g, g \rangle. \tag{6}$$

In light of Theorem 5.1 and Proposition 5.3, it only remains to calculate the ratio of $\langle \phi, \phi \rangle$ and $\langle \mathcal{V}_M \phi, \mathcal{V}_M \phi \rangle$ for $\phi \in J_{\kappa,1}^{c}(\Gamma_0(M)^J)$. We follow the arguments used in [14] where this ratio is computed when $M = 1$. One should note this inner product was originally given in [3]. However, that result cited a theorem in [6] which in turn was based on the incorrect definition of the $\mathcal{V}_M$ map used in [19]. Thus, we give the computation here using the corrected definition given in [10]. The argument given in [3, § 4] is correct up until the point the result of [6] is invoked, however we include the complete argument here to have it given in one place.

Let $F, G \in S_\kappa(\Gamma_0^{(2)}(M))$ be eigenforms with Fourier-Jacobi expansions given by

$$F(Z) = \sum_{N \geq 1} \phi_N(\tau, z) e(N\tau')$$

and

$$G(Z) = \sum_{N \geq 1} \psi_N(\tau, z) e(N\tau').$$

Define a Dirichlet series attached to $F$ and $G$ by

$$D_{F,G}(s) = \zeta^M(2s - 2\kappa + 4) \sum_{N \geq 1} \langle \phi_N, \psi_N \rangle N^{-s}$$

and set

$$D^*_{F,G}(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-\kappa+2)D_{F,G}(s). \tag{7}$$

It is shown in [9] that $D^*_{F,G}(s)$ has meromorphic continuation to $\mathbb{C}$, is entire if $\langle F, G\rangle = 0$ and otherwise has a simple pole at $s = \kappa$. The first step in calculating the ratio of inner products we desire is calculating the residue of $D_{F,G}$ at $s = \kappa$. We do this by writing $D_{F,G}$ as the Petersson product of $F(Z)\overline{G(Z)}|Y|^\kappa$ against a certain non-holomorphic Klingen Eisenstein series $E_{s,M}(Z)$. Define a Klingen Eisenstein series

$$E_{s,M}(Z) = \sum_{\gamma\in C_{2,1}(M)\backslash\Gamma_0^{(2)}(M)} \left(\frac{\det(\mathrm{Im}\gamma Z)}{\mathrm{Im}(\gamma Z)_1}\right)^s$$

where $(\gamma Z)_1$ denotes the upper left entry of $\gamma Z$ and

$$C_{2,1}(M) = \left\{ \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_0^4(M) \right\},\ (\lambda', \mu') = (\lambda, \mu)\begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

Set

$$E^*_{s,M}(Z) = \pi^{-s}\Gamma(s)\zeta(2s)\prod_{p\mid M}(1-p^{-2s})E_{s,M}(Z). \tag{8}$$

One has that $E^*_{s,M}(Z)$ has mermomorphic continuation to $\mathbb{C}$ with possible simple poles at $s = 0, 2$ ([9]). It is known that $\operatorname*{res}_{s=2} E^*_{s,1}(Z) = 1$ ([14]). Note that this is independent of $Z$, so we have $\operatorname*{res}_{s=2} E^*_{s,1}(NZ) = 1$ for all positive integers $N$. Equation (8) gives $\operatorname*{res}_{s=2} E_{s,1}(Z) = \frac{90}{\pi^2}$. As above, this residue is independent of $Z$ so we have $\operatorname*{res}_{s=2} E_{s,1}(NZ) = \frac{90}{\pi^2}$ for all positive integers $N$. The following formula is given in [9]:

$$E_{s,1}(MZ) = \frac{1}{M^s}\sum_{d\mid M} d^{2s}\prod_{p\mid d}(1-p^{-2s})E_{s,d}(Z).$$

This formula allows one to calculate the residue of $E_{s,M}(Z)$ inductively in terms of $E_{s,d}(Z)$ for $d \mid M$. In fact, for $M = p_1^{m_1}\dots p_n^{m_n}$, we have

$$\operatorname*{res}_{s=2} E_{s,M}(Z) = \left(\frac{90}{\pi^2}\right) h(p_1,\dots,p_n)\prod_{i=1}^n\left(\frac{1}{p_i^{2m_i-2}(p_i^4-1)}\right) \tag{9}$$

where $h$ is a polynomial with coefficients in $\mathbb{Z}$ uniquely determined by $M$. For example, if $M = p^n$ for a prime $p$, we have

$$h(p) = p^2 - 1$$

and if $M = p_1 \dots p_n$ is a product of distinct primes, we have

$$h(p_1, \dots, p_n) = \prod_{i=1}^{n} (p_i^2 - 1).$$

We will be mainly interested in the case that $M = p_1 \dots p_n$ is odd and square-free. Appealing to (8) we obtain

$$\operatorname*{res}_{s=2} E^*_{s,M}(Z) = \prod_{i=1}^{n} \left( \frac{1 - p_i^{-4}}{p_i^2 + 1} \right). \tag{10}$$

We now turn our attention back to calculating the residue of $D_{F,G}(s)$ at $s = \kappa$. We have the following equation (see [9]):

$$\pi^{-\kappa+2} [\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)] \langle F E^*_{s-\kappa+2,M}, G \rangle = M^s D^*_{F,G}(s).$$

Taking the residue of this equation at $s = \kappa$ and solving for $\operatorname*{res}_{s=\kappa} D^*_{F,G}(s)$ we obtain

$$\begin{aligned} \operatorname*{res}_{s=\kappa} D^*_{F,G}(s) &= \frac{\pi^{2-\kappa} \left[ \mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M) \right]}{M^\kappa} \operatorname*{res}_{s=2} E^*_{s,M}(Z) \langle F, G \rangle \\ &= \frac{\pi^{2-\kappa} \left[ \mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M) \right]}{M^\kappa} \prod_{p \mid M} \left( \frac{1 - p_i^{-4}}{p_i^2 + 1} \right) \langle F, G \rangle \\ &= \frac{\pi^{2-\kappa} \left[ \mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M) \right]}{M^\kappa \zeta_M(4)} \prod_{p \mid M} \left( \frac{1}{p_i^2 + 1} \right) \langle F, G \rangle. \end{aligned}$$

On the other hand, taking the residue at $s = \kappa$ of (7) we have

$$\operatorname*{res}_{s=\kappa} D^*_{F,G}(s) = (2\pi)^{-2\kappa} (\kappa - 1)! \operatorname*{res}_{s=\kappa} D_{F,G}(s).$$

Combining these two results and solving for $\operatorname*{res}_{s=\kappa} D_{F,G}(s)$ we obtain

$$\operatorname*{res}_{s=\kappa} D_{F,G}(s) = \frac{2^{2\kappa} \pi^{\kappa+2} \left[ \mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M) \right]}{M^\kappa \zeta_M(4) (\kappa - 1)! \prod_{p \mid M} (p_i^2 + 1)} \langle F, G \rangle. \tag{11}$$

Following [14], our next step is to calculate the adjoint of the operator $V_m$. We will need the following lemma.

**Lemma 5.4.** *Let* $\Delta^*_{M,0}(m) \subset \Delta_{M,0}(m)$ *be the matrices* $\begin{pmatrix} a & b \\ Mc & d \end{pmatrix}$ *with* $\gcd(a, b, c, d) = 1$*. The map*

$$\begin{aligned} \varphi_M : \Gamma_0(M, m) \backslash \Gamma_0(M) &\to \Gamma_0(M) \backslash \Delta^*_{M,0}(m) \\ \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \end{aligned}$$

*gives a bijection where*

$$\Gamma_0(M,m) = \left\{ \begin{pmatrix} a & mb \\ Mc & d \end{pmatrix} \in \Gamma_0(M) \right\}.$$

*Proof.* The proof amounts to showing the cardinality of $\Gamma_0(M,m)\backslash\Gamma_0(M)$ is the same as the cardinality of $\Gamma_0(M)\backslash\Delta^*_{M,0}(m)$ and then showing the map $\varphi_M$ is injective by using the explicit coset representatives given for $\Gamma_0(M,m)\backslash\Gamma_0(M)$. Write $M = \prod_{i=1}^r p^{e_i}$ and $m = \prod_{i=1}^r p^{f_i}$ where the $e_i$ and $f_i$ are nonnegative integers. One has that

$$\#\left(\Gamma_0(M)\backslash\Delta_{M,0}(m)\right) = \prod_{i=1}^r \#\left(\Gamma_0(M)\backslash\Delta_{M,0}(p_i^{f_i})\right)$$

and so

$$\#\left(\Gamma_0(M)\backslash\Delta^*_{M,0}(m)\right) = \prod_{i=1}^r \#\left(\Gamma_0(M)\backslash\Delta^*_{M,0}(p_i^{f_i})\right).$$

Thus, we only need to calculate

$$\#\left(\Gamma_0(M)\backslash\Delta^*_{M,0}(p^f)\right)$$

for a prime $p$. This breaks into two cases depending upon whether $p$ divides $M$ or not. The main input is the fact that

$$\Gamma_0(M)\backslash\Delta_{M,0}(m) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = m, \gcd(a,M) = 1, 0 \le b \le d-1 \right\}.$$

Using this, one sees that if $p \mid M$ then

$$\Gamma_0(M)\backslash\Delta^*_{M,0}(p^f) = \left\{ \begin{pmatrix} 1 & b \\ 0 & p^f \end{pmatrix} : 0 \le b < p^f \right\},$$

and so there are $p^f$ elements. If $p \nmid M$, one shows by induction on $f$ and counting as above that

$$\#\left(\Gamma_0(M)\backslash\Delta^*_{M,0}(p^f)\right) = p^f + p^{f-1}.$$

Recall the map

$$\lambda_T : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/T\mathbb{Z})$$

is surjective with kernel $\Gamma(T)$. From this, one sees that there is a bijection between $\Gamma_0(M,m)\backslash\Gamma_0(M)$ and $\lambda_T(\Gamma_0(M,m))\backslash\lambda_T(\Gamma_0(M))$ where $T = \mathrm{lcm}(M,m)$. Moreover, one has

$$\mathrm{SL}_2(\mathbb{Z}/T\mathbb{Z}) \cong \prod_{p^f \| T} \mathrm{SL}_2(\mathbb{Z}/p^f\mathbb{Z}).$$

Thus, one only needs to work with prime powers to compute the cosets. Write $M = p^e$ and $m = p^f$ for $e, f$ nonnegative integers. We break into cases for this:

(1) Suppose $e = 0$. In this case we have $\lambda_T(\Gamma_0(M,m))\backslash\lambda_T(\Gamma_0(M)) = \lambda_T(\Gamma^0(m))\backslash\lambda_T(\mathrm{SL}_2(\mathbb{Z}))$. In this case the coset representatives are given by $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for $0 \leq b < p^f$ and $\begin{pmatrix} 1 & 1 \\ (1+b)p-1 & (1+b)p \end{pmatrix}$ for $0 \leq b < p^{f-1}$ (see [7, Section 3.7]).
(2) Suppose $e$ is positive. In this case the representatives are given by $\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : 0 \leq b < p^f \right\}$.

Thus, we have the cardinalities matching up and the map $\varphi_M$ is clearly injective. □

**Proposition 5.5.** *Let $V_m^* : J_{\kappa,m}^c(\Gamma_0(M)^J) \to J_{\kappa,1}^c(\Gamma_0(M)^J)$ be the adjoint of $V_m$ with respect to the Petersson inner product. Let $\psi \in J_{\kappa,m}^c(\Gamma_0(M)^J)$ with*

$$\psi(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2(4m)}} c(D,r) e\left(\frac{r^2 - D}{4m}\tau + rz\right).$$

*The action of $V_m^*$ on Fourier coefficients is given by*

$$V_m^*\psi(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2(4)}} \left( \sum_{\substack{d\mid m \\ \gcd(d,M)=1}} d^{\kappa-2} \sum_{s\in S(r,d,D)} c\left(\frac{m^2}{d^2}D, \frac{m}{d}s\right)\right) e\left(\frac{r^2-D}{4}\tau + rz\right)$$

*where $S(r,d,D) = \{s \pmod{2d} : s \equiv r \pmod 2, s^2 \equiv D \pmod{4d}\}$.*

*Proof.* The proof here is analogous to the one given in [14] for the level 1 case. We include a proof for $M \geq 1$ and square-free here with more details for the reader's convenience. Let $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$. Given $a \in \mathbb{C}$, write $\phi_a(\tau, z)$ for the function $\phi(\tau, az)$. If $m' \mid m$ we write $m/m' = \square$ to denote that $m/m'$ is a perfect square. One has immediately from the definition and the lemma above that

$$\begin{aligned} V_m\phi &= m^{\kappa/2-1} \sum_{g\in\Gamma_0(M)\backslash\Delta_{M,0}(m)} \phi_{\sqrt{m}}|_{\kappa,m}\left(\frac{g}{\sqrt{m}}\right) \\ &= m^{\kappa/2-1} \sum_{\substack{m'\mid m \\ m/m'=\square}} \sum_{g\in\Gamma_0(M)\backslash\Delta_{M,0}^*(m)} \phi_{\sqrt{m}}|_{\kappa,m}\left(\frac{g}{\sqrt{m'}}\right) \\ &= m^{\kappa/2-1} \sum_{\substack{m'\mid m \\ m/m'=\square}} \sum_{g\in\Gamma_0(M,m)\backslash\Gamma_0(M)} \phi_{\sqrt{m}}|_{\kappa,m}\begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix} g. \end{aligned}$$

Note that $\phi_{\sqrt{m}}|_{\kappa,m}\begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix} \in J_{\kappa,m}^c(\Gamma_0(M,m)^J)$.

Given $\phi \in J^c_{\kappa,1}(\Gamma_0(M)^J)$ and $\psi \in J^c_{\kappa,m}(\Gamma_0(M)^J)$ we have

$$\begin{aligned}\langle V_m\phi, \psi\rangle &= m^{\kappa/2-1} \sum_{\substack{m'|m \\ m/m'=\square}} \sum_{g\in\Gamma_0(M,m)\backslash\Gamma_0(M)} \left\langle \phi_{\sqrt{m}}|_{\kappa,m} \begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'}\end{pmatrix} g, \psi \right\rangle \\ &= m^{\kappa/2-1} \sum_{\substack{m'|m \\ m/m'=\square}} [\Gamma_0(M):\Gamma_0(M,m)] \left\langle \phi_{\sqrt{m}}|_{\kappa,m} \begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'}\end{pmatrix}, \psi \right\rangle\end{aligned}$$

where we have used $\langle \phi|\gamma, \psi\rangle = \langle \phi, \psi|\gamma^{-1}\rangle$ and $\psi|_{\kappa,m}g = \psi$ as $\psi$ has level $\Gamma_0(M)^J$. Note that $\psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\begin{pmatrix}\sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}}\end{pmatrix} \in J^c_{\kappa,1}(\Gamma_0(M,m)^J)$. Moreover, we have

$$\left\langle \phi_{\sqrt{m}}|_{\kappa,m} \begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'}\end{pmatrix}, \psi \right\rangle = \left\langle \phi, \psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\begin{pmatrix}\sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}}\end{pmatrix}\right\rangle.$$

Observe we can write

$$\begin{aligned}&\left\langle \phi_{\sqrt{m}}|_{\kappa,m} \begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'}\end{pmatrix}, \psi \right\rangle = \left\langle \phi, \psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\begin{pmatrix}\sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}}\end{pmatrix}\right\rangle \\ &= \frac{1}{m'^2[\Gamma_0(M):\Gamma_0(M,m')]} \sum_{X\in(\mathbb{Z}/m'\mathbb{Z})^2} \sum_{g\in\Gamma_0(M,m')\backslash\Gamma_0(M)} \left\langle \phi, \psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\begin{pmatrix}\sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}}\end{pmatrix} gX\right\rangle\end{aligned}$$

Thus, by essentially reversing the above argument we obtain

$$\langle V_m\phi,\psi\rangle = \left\langle \phi, m^{\kappa/2-3} \sum_{X\in(\mathbb{Z}/m\mathbb{Z})^2} \sum_{g\in\Gamma_0(M,m)\backslash\Gamma_0(M)} \psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\left(\frac{gX}{\sqrt{m}}\right)\right\rangle.$$

One then checks that in fact

$$m^{\kappa/2-3} \sum_{X\in(\mathbb{Z}/m\mathbb{Z})^2} \sum_{g\in\Gamma_0(M)\backslash\Delta^*_{M,0}(m)} \psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\left(\frac{gX}{\sqrt{m}}\right) \in J^c_{\kappa,1}(\Gamma_0(M)^J),$$

and so we have the formula for $V_m^*: J^c_{\kappa,m}(\Gamma_0(M)^J) \to J^c_{\kappa,1}(\Gamma_0(M)^J)$.

Thus, it just remains to compute the Fourier expansion of $V_m^*\psi$.

Let $\psi \in J^c_{\kappa,m}(\Gamma_0(M)^J)$ with Fourier expansion given by

$$\psi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2(4)}} c(D,r)e\left(\frac{r^2-D}{4m}\tau + rz\right).$$

We use the same representatives for $\Gamma_0(M)\backslash\Delta_{M,0}(m)$ as above. Thus, we have

$$V_m^*\psi(\tau,z) = m^{\kappa/2-3}\sum_{\lambda,\mu(m)}\sum_{\substack{ad=m\\ \gcd(a,M)=1\\ b(d)}}\left(\frac{\sqrt{m}}{d}\right)^{\kappa} e(\lambda^2\tau+2\lambda z)\psi\left(\frac{a\tau+b}{d},\frac{z+\lambda\tau+\mu}{d}\right)$$

$$= m^{\kappa-3}\sum_{\lambda,\mu(m)}\sum_{\substack{ad=m\\ \gcd(a,M)=1\\ b(d)}} d^{-\kappa}\sum_{\substack{D<0,r\in\mathbb{Z}\\ D\equiv r^2(4m)}} c(D,r)e\left(\left(\frac{r^2-D}{4m}\frac{a}{d}+\frac{\lambda r}{d}+\lambda^2\right)\tau\right)$$

$$\cdot e\left(\left(\frac{r}{d}+2\lambda\right)z+\frac{r^2-D}{4m}\frac{b}{d}+\frac{r\mu}{d}\right).$$

Note that we have

$$\sum_{\substack{\mu(m)\\ b(d)}} e\left(\frac{r^2-D}{4m}\frac{b}{d}+\frac{r\mu}{d}\right) = \begin{cases} md & \text{if } d\mid r \text{ and } d\mid \frac{r^2-D}{4m}\\ 0 & \text{otherwise.}\end{cases}$$

Setting $r_1 = dr$ and $D_1 = d^2D$, we have

$$V_m^*\psi(\tau,z) = m^{\kappa-2}\sum_{\lambda(m)}\sum_{\substack{d\mid m\\ \gcd\left(\frac{m}{d},M\right)=1}} d^{1-\kappa}\sum_{\substack{D_1<0,r_1\in\mathbb{Z}\\ D_1\equiv r_1^2(4m/d)}} c(d^2D_1,dr_1)$$

$$\cdot e\left(\frac{(r_1+2\lambda)^2-D_1}{4}\tau+(r_1+2\lambda)z\right).$$

Letting $r_2 = r_1+2\lambda$, we have

$$V_m^*\psi(\tau,z) = m^{\kappa-2}\sum_{\lambda(m)}\sum_{\substack{d\mid m\\ \gcd\left(\frac{m}{d},M\right)=1}} d^{1-\kappa}\sum_{\substack{D_1<0,r_2\in\mathbb{Z}\\ D_1\equiv(r_2-2\lambda)^2(4m/d)}} c(d^2D_1,d(r_2-2\lambda))$$

$$\cdot e\left(\frac{r_2^2-D_1}{4}\tau+r_2z\right).$$

We can write

$$\lambda\equiv s+\frac{m}{d}s' \pmod{m}$$

where $s$ runs over $\mathbb{Z}/(m/d)\mathbb{Z}$ and $s'$ runs over $\mathbb{Z}/d\mathbb{Z}$. We immediately have

$$d(r_2-2\lambda)\equiv d(r_2-2s)\pmod{2m},\quad D_1\equiv(r_2-2s)^2\pmod{4m/d}.$$

We now use the fact that the coefficients $c(D,r)$ depend only on the pair $(D,r)$ with $r \pmod{2m}$ and $D\equiv r^2 \pmod{4m}$ to write

$$V_m^*\psi(\tau,z) = m^{\kappa-2}\sum_{\substack{d\mid m\\ \gcd\left(\frac{m}{d},M\right)=1}} d^{2-\kappa}\sum_{s(m/d)}\sum_{\substack{D_1<0,r_1\in\mathbb{Z}\\ D_1\equiv(r_2-2s)^2(4m/d)}} c(d^2D_1,d(r_2-2s)$$

$$\cdot e\left(\frac{r_2^2-D_1}{4}\tau+r_2z\right).$$

Finally, we change variables and replace $D_1$ by $D$, $d$ by $m/d$, and $r_2 - 2s$ by $s$ to obtain

$$V_m^* \psi(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2(4)}} \left( \sum_{\substack{d|m \\ \gcd(d,M)=1}} d^{\kappa-2} \sum_{s\in S(r,d,D)} c\left(\frac{m^2}{d^2}D, \frac{m}{d}s\right)\right) e\left(\frac{r^2-D}{4}\tau + rz\right)$$

where $S(r,d,D) = \{s \pmod{2d} : s \equiv r \pmod 2, s^2 \equiv D \pmod{4d}\}$. □

**Proposition 5.6.** *The map* $V_m^* V_m : J_{\kappa,1}^c(\Gamma_0(M)^J) \to J_{\kappa,1}^c(\Gamma_0(M)^J)$ *is given by*

$$V_m^* V_m = \sum_{\substack{d|m \\ \gcd(d,M)=1}} \varsigma(d) d^{\kappa-2} T_J\left(\frac{m}{d}\right)$$

*where* $T_J(n)$ *is the* $n$*th Hecke operator on* $J_{\kappa,1}^c(\Gamma_0(M)^J)$ *(we write* $U_J(p)$ *for* $T_J(p)$ *if* $p \mid M$*) and*

$$\varsigma(d) = d\prod_{p|d}\left(1+\frac{1}{p}\right).$$

*Proof.* The proof of this proposition follows along the same lines as the proof of the analogous result in the $M = 1$ case given in [14]. Note it is enough to check this fact on Fourier coefficients indexed by fundamental discriminants, as is pointed out in [14]. We show this for a representative case that is not too computationally cumbersome, but leave the proof of the general case to the reader.

Let $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$ and put $\psi = V_m\phi$, $\varphi = V_m^*\psi$. Write $c_\phi$ for the Fourier coefficients of $\phi$, $c_\psi$ for the Fourier coefficients of $\psi$, and $c_\varphi$ for the Fourier coefficients of $\varphi$.

We begin by recalling the definition of the action of $T_J(p)$ and $U_J(p)$ on the Fourier coefficients indexed by fundamental discriminants. Let $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$ with

$$\phi(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2(4)}} c_\phi(D,r) e\left(\frac{r^2-D}{4}\tau + rz\right).$$

Then we have

$$c_{U_J(p)\phi}(D,r) = c_\phi(p^2D, pr)$$

and

$$c_{T_J(p)\phi}(D,r) = c_\phi(p^2D, pr) + \chi_D(p)p^{\kappa-2}c_\phi(D,r)$$

where $\chi_D$ is the quadratic character associated to the fundamental discriminant $D$.

Consider the case where $m = pq$ with $p \mid M$, $q \nmid M$. In this case we have

$$\sum_{\substack{d|m \\ \gcd(d,M)=1}} \varsigma(d) d^{\kappa-2} T_J\left(\frac{m}{d}\right) = T_J(m) + q^{\kappa-2}(q+1)U_J(p)$$

$$= T_J(q)U_J(p) + q^{\kappa-2}(q+1)U_J(p).$$

We have that the $(D,r)$th Fourier coefficient of $(T_J(m)+q^{\kappa-2}(q+1)U_J(p))\phi$ is given by

$$c_\phi(m^2D, mr) + q^{\kappa-2}(1+q+\chi_D(q))c_\phi(p^2D, pr).$$

We now calculate the $(D,r)$th Fourier coefficient of $\varphi$. We have

$$c_\psi(D,r) = c_\phi(D,r) + q^{\kappa-1}c_\phi\left(\frac{D}{q^2}, \frac{r}{q}\right).$$

Using the above formula for $V_m^*$ on the Fourier coefficients we have

$$c_\varphi(D,r) = \sum_{\substack{d|m \\ \gcd(d,M)=1}} d^{\kappa-2} \sum_{s\in S(r,d,D)} c_\psi\left(\frac{m^2}{d^2}D, \frac{m}{d}s\right)$$

$$= \sum_{s\in S(r,1,D)} c_\psi(m^2D, ms) + q^{\kappa-2} \sum_{s\in S(r,q,D)} c_\psi(p^2D, ps).$$

We have

$$\sum_{s\in S(r,1,D)} c_\psi(m^2D, ms) = c_\psi(m^2D, mr)$$

$$= c_\phi(m^2D, mr) + q^{\kappa-1}c_\phi(p^2D, pr).$$

For the second summation we have

$$\sum_{s\in S(r,q,D)} c_\psi(p^2D, ps) = \sum_{s\in S(r,q,D)} \left(c_\phi(p^2D, ps) + q^{\kappa-1}c_\phi\left(\frac{p^2D}{q^2}, \frac{ps}{q}\right)\right)$$

$$= \sum_{s\in S(r,q,D)} c_\phi(p^2D, ps)$$

where we have used that $c_\phi(x,y) = 0$ unless $x$ and $y$ are both integers and for $\frac{p^2D}{q^2}$ to be an integer we must have $q^2 \mid D$, which cannot happen because $D$ is assumed to be a fundamental discriminant so cannot be divisible by the square of a prime. We have

$$\sum_{s\in S(r,q,D)} c_\phi(p^2D, ps) = c_\phi(p^2D, pr) \sum_{s\in S(r,q,D)} 1$$

$$= (1+\chi_D(q))c_\phi(p^2D, pr)$$

as the sum is counting whether $D$ is a square modulo $q$ or not. Thus, combining these we have

$$c_\varphi(D,r) = c_\phi(m^2D, mr) + q^{\kappa-2}\left(1+q+\chi_D(q)\right)c_\phi(p^2D, pr),$$

which is exactly what we were trying to prove. □

Let $F = \mathcal{V}_M\phi$ for $\phi \in J^c_{\kappa,1}(\Gamma_0(M)^J)$. Then we have

$$D_{F,F}(s) = \zeta^M(2s-2\kappa+4)\sum_{m\geq 1}\langle V_m\phi, V_m\phi\rangle m^{-s}.$$

We have from the previous proposition that

$$\begin{aligned}\langle V_m\phi, V_m\phi\rangle &= \langle V_m^* V_m\phi, \phi\rangle \\ &= \left\langle \sum_{\substack{d|m \\ \gcd(d,M)=1}} \varsigma(d) d^{\kappa-2} T_J\left(\frac{m}{d}\right)\phi, \phi\right\rangle \\ &= \sum_{\substack{d|m \\ \gcd(d,M)=1}} \varsigma(d) d^{\kappa-2}\lambda_f\left(\frac{m}{d}\right)\langle\phi,\phi\rangle\end{aligned}$$

where we recall that $T_J(n)\phi = \lambda_f(n)\phi$. Thus, we have

$$D_{F,F}(s) = \zeta^M(2s-2\kappa+4)\langle\phi,\phi\rangle\sum_{m\geq 1}\left(\sum_{\substack{d|m \\ \gcd(d,M)=1}} \varsigma(d) d^{\kappa-2}\lambda_f\left(\frac{m}{d}\right)\right) m^{-s}.$$

If we set

$$\begin{aligned}A(s) &= \sum_{\substack{d\geq 1 \\ \gcd(d,M)=1}} a(d) d^{-s}, \\ B(s) &= \sum_{t\geq 1} b(t) t^{-s}\end{aligned}$$

and

$$C(s) = \sum_{m\geq 1}\left(\sum_{\substack{dt=m \\ \gcd(d,M)=1}} a(d)b(t)\right) m^{-s},$$

then we have

$$C(s) = A(s)B(s).$$

We can apply this with $a(d) = \varsigma(d)d^{\kappa-2}$ and $b(t) = \lambda_f(t)$ to obtain

$$\begin{aligned}D_{F,F}(s) &= \zeta^M(2s-2\kappa+4)\langle\phi,\phi\rangle\left(\sum_{\substack{d\geq 1 \\ \gcd(d,M)=1}} \varsigma(d) d^{-s+\kappa-2}\right)\left(\sum_{t\geq 1}\lambda_f(t)t^{-s}\right) \\ &= \zeta^M(2s-2\kappa+4)\langle\phi,\phi\rangle L(s,f)\left(\sum_{\substack{d\geq 1 \\ \gcd(d,M)=1}} \varsigma(d) d^{-s+\kappa-2}\right).\end{aligned}$$

One can check immediately by expanding the right hand side that

$$\sum_{\substack{d\geq 1\\ \gcd(d,M)=1}} \varsigma(d)d^{-s} = \frac{\zeta^M(s-1)\zeta^M(s)}{\zeta^M(2s)}.$$

Thus, we have

$$D_{F,F}(s) = \zeta^M(s-\kappa+1)\zeta^M(s-\kappa+2)\langle\phi,\phi\rangle L(s,f).$$

In particular, taking the residue of each side at $s=\kappa$ we obtain

$$\begin{aligned}\operatorname*{res}_{s=\kappa} D_{F,F}(s) &= \operatorname*{res}_{s=1} \zeta^M(s)\zeta^M(2)\langle\phi,\phi\rangle L(\kappa,f)\\ &= \frac{\pi^2}{6}\left(\prod_{p\mid M}(1-p^{-1})^2(1+p^{-1})\right)\langle\phi,\phi\rangle L(\kappa,f).\end{aligned}$$

We now combine this with equations (1), (6), and (11) to obtain the following corollary.

**Corollary 5.7.** *Let $\kappa\geq 2$ be an even integer, $M$ an odd square-free integer, and $f\in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(M))$ a newform. Let $F_f\in S_\kappa(\Gamma_0^{(2)}(M))$ be the Saito-Kurokawa lift of $f$. Then we have*

$$\langle F_f,F_f\rangle = \mathcal{A}_{\kappa,M}\frac{\left|a_{\theta^{\mathrm{alg}}_{\kappa,D}(f)}(|D|)\right|^2}{|D|^{\kappa-3/2}}\frac{L(\kappa,f)}{\pi L(\kappa-1,f,\chi_D)}\langle f,f\rangle$$

*where*

$$\mathcal{A}_{\kappa,M} = \frac{M^\kappa\zeta_M(4)\zeta_M(1)^2(\kappa-1)\left(\prod_{p\mid M}(1+p^2)(1+p^{-1})\right)}{2^{\nu(M)+3}[\Gamma_0(M):\Gamma_0(4M)][\mathrm{Sp}_4(\mathbb{Z}):\Gamma_0^{(2)}(M)]}.$$

One should note one can give a similar expression for general odd $M$ that depends upon the function $h(p_1,\ldots,p_n)$ that shows up in the calculation of the residue of the Eisenstein series $E_{s,M}(Z)$ above. As we will only be interested in the case of $M$ odd and square-free, we restrict our attention to this case.

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