SPECIAL VALUES OF L-FUNCTIONS ON $\mathrm{GSp}_4 \times \mathrm{GL}_2$ AND THE NON-VANISHING OF SELMER GROUPS

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ABSTRACT. In this paper we show how one can use an inner product formula of Heim giving the inner product of a pullback Eisenstein series from Sp_{10} to $\mathrm{Sp}_2 \times \mathrm{Sp}_4 \times \mathrm{Sp}_4$ with a newform on GL_2 and a Saito-Kurokawa lift to produce congruences between Saito-Kurokawa lifts and non-CAP forms. This congruence is in part controlled by the L-function on $\mathrm{GSp}_4 \times \mathrm{GL}_2$. The congruence is then used to produce nontrivial torsion elements in an appropriate Selmer group, providing evidence for the Bloch-Kato conjecture.

1. Introduction

The Bloch-Kato conjecture for modular forms roughly states that given a newform f, the special values of the L-function attached to f should measure the size of the Selmer group associated to twists of the Galois representation ρ_f. In this paper we provide evidence for this conjecture in the form of showing that under certain hypotheses if $p \mid L_{\mathrm{alg}}(k, f)$, then $p \mid \# \mathrm{Sel}(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$. For a precise statement of this result see Theorem 10.4.

The argument used to prove Theorem 10.4 is in the spirit of the work of Ribet ([R76]) and Wiles ([Wi90]). For more recent examples of such arguments the reader is advised to consult ([JB2], [KK07], [SU06]). We now give a brief outline of the argument.

Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a newform and F_f the Saito-Kurokawa lift of f. Our first step is to establish a congruence between F_f and a non-CAP cuspidal eigenform. Let $E_{10}(Z)$ be the Siegel Eisenstein series on Sp_{10}. We pull E_{10} back to $\mathrm{Sp}_2 \times \mathrm{Sp}_4 \times \mathrm{Sp}_4$. One of the main inputs into the congruence is an inner product relation of Heim ([H99]) which states

$$\langle\langle\langle E_{10}(\mathrm{diag}[Z_1, Z_2, Z_3]), h\rangle F_f\rangle F_f\rangle = \mathcal{A}\frac{L_{\mathrm{alg}}(2k-4, f) L_{\mathrm{alg}}(2k-3, F_f \times h)}{L_{\mathrm{alg}}(k, f)}$$

where $\mathcal{A}$ is an explicit value we suppress here to ease the notation, see Corollary 5.2 for the precise value of $\mathcal{A}$. This inner product relation is analogous to the inner product relation of Shimura used in [JB2] to produce a congruence. The difficulty here is that where the Eisenstein series used in [JB2] pulls back to a form that is cuspidal in each variable ([JB1]), in this

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case the Eisenstein series is of level one so does not pull back to something cuspidal. What this means for our purposes is that the formula is not as clean and easy to work with. In an attempt to deal with this we act on E_{10} by several Hecke operators to remove as many of the extraneous pieces as possible. Ultimately, under certain hypotheses we are able to produce a non-CAP cuspidal eigenform G so that the eigenvalues of G are congruent to those of F_f.

Once the congruence has been established, arguments using the 4-dimensional p-adic Galois representation attached to G are used to produce a nontrivial p-torsion element in the Selmer group.

Future work in this direction will aim to generalize Heim's result for levels greater then 1. Such a generalization would allow one greater freedom in choosing h, allowing one to remove some of the hypotheses forced here. In particular, one should be able to choose h so that the L-values related to h showing up in the hypotheses are p-units.

2. Notation

In this section we set the notation and definitions to be used throughout this paper.

Let $\mathbb{A}$ be the adeles over $\mathbb{Q}$. For a prime p, we fix once and for all compatible embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. We denote by ε_p the p-adic cyclotomic character $\varepsilon_p : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_1(\mathbb{Z}_p)$. We drop the p when it is clear from context. We denote the composition of ε_p with the natural map $\mathrm{GL}_1(\mathbb{Z}_p) \rightarrow \mathrm{GL}_1(\mathbb{F}_p)$ by ω_p, again dropping the p when it is clear from context.

For a ring R, we let $\mathrm{M}_n(R)$ denote the ring of n by n matrices with entries in R. For a matrix $x \in \mathrm{M}_{2n}(R)$, we write

$$x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix}$$

where a_x, b_x, c_x, and d_x are all in $\mathrm{M}_n(R)$. We drop the subscript x when it is clear from the context. The transpose of a matrix x is denoted by ${}^t x$.

Let SL_n and GL_n have their standard definitions. We denote the complex upper half-plane by $\mathfrak{h}^1$. We have the usual action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathfrak{h}^1 \cup \mathbb{P}^1(\mathbb{Q})$ given by linear fractional transformations, namely, given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and $z \in \mathfrak{h}^1$, one has

$$\gamma z = \frac{az+b}{cz+d}.$$

Define

$$\mathrm{Sp}_{2n} = \{\gamma \in \mathrm{GL}_{2n} : {}^t\gamma \iota_{2n} \gamma = \iota_{2n}\}, \quad \iota_{2n} = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}.$$

Siegel upper half-space is defined by

$$\mathfrak{h}^n = \{Z \in \mathrm{M}_n(\mathbb{C}) : {}^tZ = Z, \mathrm{Im}(Z) > 0\}.$$

The group $\mathrm{Sp}_{2n}(\mathbb{R})$ acts on $\mathfrak{h}^n$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ+B)(CZ+D)^{-1}.$$

We let $\Gamma_1^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ be the Jacobi modular group. As the Jacobi modular group will not play a major role in this paper we refer the reader to ([EZ85]) for more details.

Given an L-function $L(s) = \prod_p L_p(s)$ and a finite set of places Σ, we write

$$L^{\Sigma}(s) = \prod_{p \notin \Sigma} L_p(s)$$

when we restrict to places away from Σ and

$$L_{\Sigma}(s) = \prod_{p \in \Sigma} L_p(s)$$

when we restrict to the places in Σ.

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. We write $M_k(\Gamma)$ to denote the space of modular forms of weight k and level Γ. We let $S_k(\Gamma)$ denote the subspace of cusp forms. The nth Fourier coefficient of $f \in M_k(\Gamma)$ is denoted by $a_f(n)$. Given a modular form f, we write f^c to denote the modular forms with Fourier coefficients the complex conjugates of the the Fourier coefficients of f. Given a ring $R \subset \mathbb{C}$, we write $M_k(\Gamma, R)$ for the space of modular forms with Fourier coefficients in R and similarly for $S_k(\Gamma, R)$. Let $f_1, f_2 \in M_k(\Gamma)$ with at least one of the f_i a cusp form. The Petersson product is given by

$$\langle f_1, f_2 \rangle = \frac{1}{[\overline{\mathrm{SL}_2(\mathbb{Z})} : \overline{\Gamma}]} \int_{\Gamma \backslash \mathfrak{h}^1} f_1(z) \overline{f_2(z)} y^{k-2} dx dy$$

where $\overline{\mathrm{SL}_2(\mathbb{Z})} = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1_2\}$ and $\overline{\Gamma}$ is the image of Γ in $\overline{\mathrm{SL}_2(\mathbb{Z})}$. The nth Hecke operator $T(n)$ has its usual meaning. Let $\mathbb{T}_{\mathbb{Z}}$ be the $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_k(\mathrm{SL}_2(\mathbb{Z}))$ generated by $T(n)$ for $n = 1, 2, 3, \dots$. Note that we do not include the weight in the notation as it will always be clear from context. Let A be a $\mathbb{Z}$-algebra. We set $\mathbb{T}_A = \mathbb{T}_{\mathbb{Z}} \otimes_{\mathbb{Z}} A$. We say $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ is a newform if it is an eigenform for all $T(n)$ and $a_f(1) = 1$. The L-function associated to a newform f of weight k is given by

$$L(s, f) = \sum_{n \geq 1} a_f(n) n^{-s}.$$

The L-function $L(s, f)$ has an Euler product given by

$$L(s, f) = \prod_p (1 - a_f(p) p^{-s} + p^{k-1-2s})^{-1}.$$

The Euler product can be factored as

$$L(s,f)=\prod_p[(1-\alpha_f(p)p^{-s})(1-\beta_f(p)p^{-s})]^{-1}$$

where $\alpha_f(p)+\beta_f(p)=a_f(p)$ and $\alpha_f(p)\beta_f(p)=p^{k-1}$. The terms $\alpha_f(p)$ and $\beta_f(p)$ are referred to as the pth Satake parameters of f. Let $h\in S_l(\mathrm{SL}_2(\mathbb{Z}))$ be a newform of weight l. Using the Satake parameters of $L(s,f)$ and $L(s,h)$ we define the Rankin L-function associated to f and h by

$$L(s,f\times h)=\prod_p[(1-\alpha_f(p)\alpha_h(p))(1-\alpha_f(p)\beta_h(p))(1-\beta_f(p)\alpha_h(p))(1-\beta_f(p)\beta_h(p))]^{-1}.$$

Kohnen's $+$-space of half-integral weight modular forms is given by

$$S^+_{k-1/2}(\Gamma_0(4))=\{g\in S_{k-1/2}(\Gamma_0(4)):a_g(n)=0\text{ if }(-1)^{k-1}n\equiv 2,3(\operatorname{mod}4)\}.$$

The Petersson product on $S^+_{k-1/2}(\Gamma_0(4))$ is given by

$$\langle g_1,g_2\rangle=\int_{\Gamma_0(4)\backslash\mathfrak{h}^1}g_1(z)\overline{g_2(z)}y^{k-5/2}dxdy.$$

We denote the space of Jacobi cusp forms on Γ_1^{J} by $J_{k,1}^{\mathrm{cusp}}(\Gamma_1^{\mathrm{J}})$. The inner product is given by

$$\langle\phi_1,\phi_2\rangle=\int_{\Gamma_1^{\mathrm{J}}\backslash\mathfrak{h}^1\times\mathbb{C}}\phi_1(\tau,z)\overline{\phi_2(\tau,z)}v^{k-3}e^{-4\pi y^2/v}dx\,dy\,du\,dv$$

for $\phi_1,\phi_2\in J_{k,1}^{\mathrm{cusp}}(\Gamma_1^{\mathrm{J}})$ and $\tau=u+iv$, $z=x+iy$.

We denote the space of Siegel modular forms of weight k and level $\Gamma\subset\mathrm{Sp}_{2n}(\mathbb{Z})$ by $\mathcal{M}_k(\Gamma)$. The subspace of cusp forms are denoted by $\mathcal{S}_k(\Gamma)$. In the case that $n=1$ we recover the elliptic modular forms already discussed. For $F,G\in\mathcal{M}_k(\mathrm{Sp}_{2n}(\Gamma))$ with at least one a cusp form, the Petersson product is given by

$$\langle F,G\rangle=\frac{1}{[\overline{\mathrm{Sp}_{2n}(\mathbb{Z})}:\overline{\Gamma}]}\int_{\Gamma\backslash\mathfrak{h}^n}F(Z)\overline{G(Z)}\det(\mathrm{Im}(Z))^k d\mu(Z).$$

We will be particularly interested in the decomposition

$$\mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))=\mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))\oplus\mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$$

where $\mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ is the space of Maass spezialchars and $\mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$ is the orthogonal complement. A form $F\in\mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ is in $\mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ if the Fourier coefficients of F satisfy the relation

$$A_F(n,r,m)=\sum_{d\mid\gcd(n,r,m)}d^{k-1}A_F\left(\frac{nm}{d^2},\frac{r}{d},1\right).$$

We let $T^S(n)$ denote the nth Siegel Hecke operator. As above, we set $\mathbb{T}_{\mathbb{Z}}^S$ to be the $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{S}_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ generated by the $T(n)$. For a $\mathbb{Z}$-algebra A we write $\mathbb{T}_A^S=\mathbb{T}_{\mathbb{Z}}^S\otimes_{\mathbb{Z}}A$. The Hecke algebra $\mathbb{T}_{\mathbb{C}}^S$ respects the decomposition of $\mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ into the space of Maass and non-Maass forms ([A80]).

Let $F, G \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}), \mathcal{O})$ for $\mathcal{O}$ the ring of integers of a finite extension $E/\mathbb{Q}_p$. Let ϖ be the uniformizer of E and $\mathbb{F}$ the residue field. We write $F \equiv G (\mathrm{mod}\, \varpi^m)$ for some $m \geq 1$ if $\mathrm{ord}_{\varpi}(A_F(T) - A_G(T)) \geq m$ for all T, i.e., if we have a congruence between the Fourier coefficients of F and G modulo ϖ^m. If F and G are Hecke eigenforms and we have a congruence between the eigenvalues of F and G we write $F \equiv_{\mathrm{ev}} G (\mathrm{mod}\, \varpi^m)$.

Let $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform with eigenvalues $\lambda_F(m)$. Associated to F is an L-function called the spinor L-function. It is defined by

$$L_{\mathrm{spin}}(s, F) = \zeta(2s - 2k + 4) \sum_{m \geq 1} \lambda_F(m) m^{-s}.$$

One can also define the spinor L-function in terms of the Satake parameters α_0, α_1, and α_2 of F. One has

$$L_{\mathrm{spin}}(s, F) = \prod_p Q_p(p^{-s})^{-1}$$

where

$$Q_p(X) = (1 - \alpha_0 X)(1 - \alpha_0 \alpha_1 X)(1 - \alpha_0 \alpha_2 X)(1 - \alpha_0 \alpha_2 \alpha_2 X).$$

Given a Hecke eigenform $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ and a newform $h \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, we define the L-function $L(s, F \times h)$ by

$$L(s, F \times h) = \prod_p [Q_p(\alpha_h(p) p^{-s}) Q_p(\beta_h(p) p^{-s})]^{-1}.$$

3. Saito-Kurokawa lifts

In this section we briefly recall the Saito-Kurokawa correspondence and relevant facts we will need. For a more thorough discussion the reader is urged to consult [EZ85], [Ge], or [Z80]. For a more detailed discussion of the facts we will need here the reader is advised to consult the section on the Saito-Kurokawa correspondence in [JB2]

Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a newform. The Saito-Kurokawa correspondence associates a cuspidal Siegel eigenform $F_f \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ to f. In the language of automorphic forms F_f is a CAP form, i.e., F_f is a cuspform that has the same eigenvalues almost everywhere as the Eisenstein series induced from the automorphic representation π_f of GL_2 associated to f when viewed on the Siegel parabolic to an automorphic representation on GSp_4. For more details on the Saito-Kurokawa correspondence from this point of view one should consult Piatetski-Shapiro's original paper on the subject ([PS83]). The Saito-Kurokawa correspondence is stated in the following theorem.

Theorem 3.1. *([Z80]) There is a Hecke-equivariant isomorphism between $S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ and $\mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ such that if $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ is a newform, then one has*

$$L_{\mathrm{spin}}(s, F_f) = \zeta(s - k + 1)\zeta(s - k + 2) L(s, f). \tag{1}$$

We will be interested in the Saito-Kurokawa correspondence from a classical point of view as constructed by a series of lifts. Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{O})$ be a newform. The first step in the Saito-Kurokawa correspondence is achieved via the Shimura-Shintani correspondence giving an isomorphism between $S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ and Kohnen's $+$-space of half-integral weight modular forms $S^+_{k-1/2}(\Gamma_0(4))$. The isomorphism is most easily expressed in terms of the image of a half-integral weight modular form $g \in S^+_{k-1/2}(\Gamma_0(4))$. The map is given by sending

$$g(z) = \sum_{\substack{n \geq 1 \\ (-1)^{k-1} n \equiv 0,1 \pmod 4}} c_g(n) q^n \in S^+_{k-1/2}(\Gamma_0(4))$$

to

$$\zeta_D g(z) = \sum_{n=1}^{\infty} \left(\sum_{d \mid n} \left(\frac{D}{d} \right) d^{k-2} c_g(|D| n^2/d^2) \right) q^n$$

where D is a fundamental discriminant with $(-1)^{k-1} D > 0$.

The second lift in the Saito-Kurokawa correspondence is an isomorphism between Kohnen's $+$-space and the space of cuspidal Jacobi forms $J^{\mathrm{cusp}}_{k,1}(\Gamma^J_1)$. The map is given by

$$\sum_{\substack{D<0, r \in \mathbb{Z} \\ D \equiv r^2 \pmod 4}} c(D,r) e\left(\frac{r^2 - D}{4} \tau + rz \right) \mapsto \sum_{\substack{D<0 \\ D \equiv 0,1 \pmod 4}} c(D) e(|D| \tau).$$

The final lift needed provides an isomorphism between the space of cuspidal Jacobi forms and the space of Maass spezialchars $\mathcal{S}^{\mathrm{M}}_k(\mathrm{Sp}_4(\mathbb{Z}))$ given by

$$\phi(\tau, z) \mapsto F(\tau, z, \tau') = \sum_{m \geq 0} V_m \phi(\tau, z) e(m\tau')$$

where V_m is the index shifting operator as defined in ([EZ85], § 4).

Our interest in the Saito-Kurokawa correspondence is in terms of arithmetic applications. As such, we will need the following three results.

Corollary 3.2. *(*[JB2]*, Corollary 3.8) Given $f \in S_k(\mathrm{SL}_2(\mathbb{Z}), \mathcal{O})$ a newform, then F_f also has Fourier coefficients in $\mathcal{O}$. In particular, if $\mathcal{O}$ is a discrete valuation ring, F_f has a Fourier coefficient in $\mathcal{O}^\times$.*

Corollary 3.3. *(*[JB1]*, Corollary 4.3) Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ and F_f the Saito-Kurokawa lift of f. One has that $F_f^c = F_{f^c}$. In other words, the Saito-Kurokawa correspondence respects complex conjugation of Fourier coefficients.*

Theorem 3.4. *(*[KS89], [KZ81]*) Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a newform, $F_f \in \mathcal{S}^{\mathrm{M}}_k(\mathrm{Sp}_4(\mathbb{Z}))$ the corresponding Saito-Kurokawa lift, and $g(z) = \sum c_g(n) q^n$*

the weight $k - 1/2$ cusp form corresponding to f under the Shintani map. We have the following inner product relation

$$\langle F_f, F_f \rangle = \frac{(k-1)}{2^5 3^2 \pi} \cdot \frac{c_g(|D|)^2}{|D|^{k-3/2}} \cdot \frac{L(k,f)}{L(k-1,f,\chi_D)} \langle f, f \rangle$$

where D is a fundamental discriminant so that $(-1)^{k-1}D > 0$ and χ_D is the character associated to the quadratic field $\mathbb{Q}(\sqrt{D})$.

Proposition 3.5. *([JB2]) Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a newform, ρ_f the associated ℓ-adic Galois representation, F_f the Saito-Kurokawa lift, and ρ_{F_f} the associated 4-dimensional ℓ-adic Galois representation. Then one has*

$$\rho_{F_f} = \begin{pmatrix} \varepsilon_\ell^{k-2} & & \\ & \rho_f & \\ & & \varepsilon_\ell^{k-1} \end{pmatrix}$$

where the blank spaces in the matrix are assumed to be 0's of the appropriate size.

4. Siegel Eisenstein series

As the Siegel Eisenstein series will be an important tool in producing the congruence between the Saito-Kurokawa lift and a form in $\mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$, we briefly recall here the definition as well as the appropriate normalization for working with arithmetic applications.

Let K_{2n} be the maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{A})$ defined by $K_{2n} = K_\infty K_f$ where

$$K_\infty = \{g \in \mathrm{Sp}_{2n}(\mathbb{R}) : g\mathbf{i}_{2n} = \mathbf{i}_{2n}\}$$

where $\mathbf{i}_{2n} = i1_{2n}$ and

$$K_f = \prod_p \mathrm{Sp}_{2n}(\mathbb{Z}_p).$$

Let

$$\mathbb{S}_n = \{x \in \mathrm{M}_n : {}^t x = x\}.$$

We denote the Siegel parabolic of Sp_{2n} by $P_{2n} = U_{2n}Q_{2n}$ where U_{2n} is the unipotent radical given by

$$U_{2n} = \left\{ u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{S}_n \right\}$$

and Q_{2n} is the Levi subgroup given by

$$Q_{2n} = \left\{ Q(A) = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} : A \in \mathrm{GL}_n \right\}.$$

We drop the subscript $2n$ on K_{2n}, P_{2n}, U_{2n}, and Q_{2n} when it is clear from context.

We are now able to define the Siegel Eisenstein series of weight $k \geq 2$. Define $\varepsilon(g, s; k)$ on $\mathrm{Sp}_{2n}(\mathbb{A}) \times \mathbb{C}$ by setting

$$\varepsilon(g, s; k) = 0$$

if $g \notin P(\mathbb{A})K$ and for $g = u(x)Q(A)\theta$ with $u(x)Q(A) \in P(\mathbb{A})$ and $\theta \in K$ we set

$$\varepsilon(g, s; k) = \varepsilon_\infty(g, s; k) \prod_\ell \varepsilon_\ell(g, s; k)$$

where the components are defined by

$$\varepsilon_\infty(g, s; k) = |\det A_\infty|^{2s} j^{-k}(\theta_\infty, \mathbf{i}),$$
$$\varepsilon_\ell(g, s; k) = |\det A_\ell|^{2s}.$$

The (adelic) Siegel Eisenstein series is defined by

$$E_{2n}(g, s; k) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{Sp}_{2n}(\mathbb{Q})} \varepsilon(\gamma g, s; k).$$

It is well-known that the series $E_{2n}(g, s; k)$ converges locally uniformly for $\mathrm{Re}(s) > (n+1)/2$ and can be continued to a meromorphic function on all of $\mathbb{C}$.

There is also a complex version of the Siegel Eisenstein series $E(Z, s; k) := E_{2n}(Z, s; k)$ defined by

$$E_{2n}(Z, s; k) = j^k(g_\infty, \mathbf{i}) E_{2n}(g, s; k)$$

where $Z = g_\infty \mathbf{i}$ and $g = g_{\mathbb{Q}} g_\infty \theta_f \in \mathrm{Sp}_{2n}(\mathbb{Q}) \mathrm{Sp}_{2n}(\mathbb{R}) K_f$. We will be interested in this complex version when $s = 0$. In this case we have

$$E_{2n}(Z) := E_{2n}(Z, 0; k) = \sum_{\gamma \in P(\mathbb{Z}) \backslash \mathrm{Sp}_{2n}(\mathbb{Z})} j^{-k}(\gamma, Z).$$

It is known that $E_{2n}(Z) \in \mathcal{M}_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$, see for example [Kl90] or [Sh87].

As our applications are to arithmetic problems, it is important to understand the Fourier coefficients of $E_{2n}(Z)$. The Fourier coefficients of $E_{2n}(Z)$ are all rational numbers. Moreover, the possible denominators of the Fourier coefficients are bounded as follows. Let d_k be the product of the numerators of $2B_k/k$ and B_{2j}/j for $j = 1, \dots, k-1$ where B_j is the j^{th} Bernoulli number. Define $z_k \in \mathbb{Z}$ by $2^{z_k} < k \leq 2^{z_k+1}$. If $k \equiv 0 \pmod 4$, then the common denominator of all the Fourier coefficients of $E_{2n}(Z)$ divides d_k and otherwise it divides $2^{z_k - 1} d_k$ ([Si64]). Consider the normalized Eisenstein series $\mathcal{E}_{2n}(Z) = 2^{z_k-1} d_k E_{2n}(Z)$ and let p be a prime. We have that $\mathcal{E}_{2n}(Z)$ is a Siegel modular form of weight k with Fourier coefficients in $\mathbb{Z}_p$, i.e., $\mathcal{E}_{2n}(Z) \in \mathcal{M}_k(\mathrm{Sp}_{2n}(\mathbb{Z}), \mathbb{Z}_p)$. This normalized Eisenstein series is the one we will work with for our arithmetic applications.

5. Pullbacks and an inner product relation

Let $(n_1, n_2, \dots, n_r)$ be a partition of n, i.e., $n = \sum_{i=1}^r n_i$. Let $\Gamma_i \subset \mathrm{Sp}_{2n_i}(\mathbb{Z})$ be congruence subgroups and $\mathbf{k} = (k_1, \dots, k_r)$ with k_i positive integers for $i = 1, \dots, r$. Set $\Gamma = \prod_{i=1}^r \Gamma_i$ and $\mathfrak{H} = \prod_{i=1}^r \mathfrak{h}^{n_i}$. Let $\mathscr{M}(\mathbf{k}, \Gamma)$ denote the space of holomorphic functions F on $\mathfrak{H}$ such that as a function on $\mathfrak{h}^{n_i}$ one has $F \in \mathcal{M}_{k_i}(\Gamma_i)$ for $i = 1, \dots, r$. Similarly one defines the space

$\mathscr{S}(\mathbf{k},\Gamma)$ as the space of holomorphic functions F on $\mathfrak{H}$ so that when viewed as a function on $\mathfrak{h}^{n_i}$ one has $F \in \mathcal{S}_{k_i}(\Gamma_i)$.

Let $F_i \in \mathcal{M}_{k_i}(\Gamma_i)$ and set $F(Z) = F_1(Z_1)F_2(Z_2)\cdots F_r(Z_r)$ where $Z = (Z_1,\dots,Z_r)$. Clearly we have that $F \in \mathscr{M}(\mathbf{k},\Gamma)$ and similarly if the F_i are cusp forms then $F \in \mathscr{S}(\mathbf{k},\Gamma)$. In fact, Lemma 1.1 of [Sh83] gives that all elements of $\mathscr{M}(\mathbf{k},\Gamma)$ are finite sums of such functions and similarly for cusp forms. Every element of $\mathscr{M}(\mathbf{k},\Gamma)$ has a Fourier expansion

$$F(Z) = \sum_T a_F(T) e^{2\pi i \operatorname{Tr}(TZ)}$$

where T runs over some lattice. The important point for us is that if $F(Z) = F_1(Z_1)F_2(Z_2)\cdots F_r(Z_r)$, then this is simply the product of the Fourier expansions of $F_i(Z_i)$. For $F, G \in \mathscr{M}(\mathbf{k},\Gamma)$ with at least one a cusp form, one defines

$$\langle F, G\rangle = \operatorname{vol}(\Gamma\backslash\mathfrak{H})^{-1}\int_{\Gamma\backslash\mathfrak{H}} F(Z)\overline{G(Z)}\det(\operatorname{Im}(Z))^k d\mu(Z).$$

One has that if $F(Z) = F_1(Z_1)F_2(Z_2)\cdots F_r(Z_r)$ and $G(Z) = G_1(Z_1)G_2(Z_2)\cdots G_r(Z_r)$ with at least one of F_i or G_i cuspidal for each $i = 1,\dots,r$, then

$$\langle F, G\rangle = \prod_{i=1}^r \langle F_i, G_i\rangle.$$

We have an embedding ι of $\mathrm{Sp}_{2n_1}\times\cdots\mathrm{Sp}_{2n_r}$ into Sp_{2n} given by

$$\iota\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1\end{pmatrix}\times\cdots\times\begin{pmatrix} a_r & b_r \\ c_r & d_r\end{pmatrix}\right) = \begin{pmatrix} a_1 & & 0 & b_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & a_r & 0 & & b_r \\ c_1 & & 0 & d_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & c_r & 0 & & d_r \end{pmatrix}.$$

As we will be working classically rather then adelically, we make use of the embedding

$$\mathfrak{h}^{n_1}\times\cdots\times\mathfrak{h}^{n_r}\hookrightarrow\mathfrak{h}^{2n}$$

given by

$$Z_1\times\cdots\times Z_r \mapsto \begin{pmatrix} Z_1 & & 0 \\ & \ddots & \\ 0 & & Z_r\end{pmatrix} = \operatorname{diag}[Z_1,\dots,Z_r]$$

arising from the isomorphism $\mathrm{Sp}_{2n}(\mathbb{R})/K_\infty \cong \mathfrak{h}^n$. We denote this map by ι as well. Given a modular form F of weight k on $\mathfrak{h}^n$, the function $F\circ\iota \in \mathscr{M}(\mathbf{k},\Gamma)$ for $\mathbf{k} = (k,\dots,k)$ and $\Gamma_i = \mathrm{Sp}_{2n_i}(\mathbb{Z})$. We refer to $F\circ\iota$ as the pullback of F from Sp_{2n} to $\mathrm{Sp}_{2n_1}\times\cdots\times\mathrm{Sp}_{2n_r}$ or just the pullback of F when the partition of n is clear from context.

In this paper we are interested in the case of $\mathrm{Sp}_2 \times \mathrm{Sp}_4 \times \mathrm{Sp}_4$ embedded in Sp_{10} and from now on we restrict ourselves to this case.. Recall that $\mathrm{Sp}_2 \cong \mathrm{SL}_2$, so we view this as an embedding $\mathrm{SL}_2 \times \mathrm{Sp}_4 \times \mathrm{Sp}_4$ into Sp_{10}. For $F_1 \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ and $F_2, F_3 \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$, we define

$$\Psi(E_{10}, F_1, F_2, F_3, s) = \langle\langle\langle E_{10} \circ \iota(Z_1, Z_2, Z_3), s; k), F_1(Z_1)\rangle, F_2(Z_2)\rangle, F_3(Z_3)\rangle.$$

Set

$$\mathcal{L}(f, G, h^c, s) = \frac{L(2s+2k-4, f)L(2s+2k-3, f)L(s+2k-3, G \times h^c)}{L(2s+2k-3, f^c)}.$$

Our interest in this iterated inner product is the following result.

Theorem 5.1. *(*[H99]*, Theorem 5.1) Let $F_f \in \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$, $G \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$, and $h \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be Hecke eigenforms. Suppose k is even and h is a newform. Then for $s \in \mathbb{C}$ with $2\,\mathrm{Re}(s) + k > 6$, we have*

$$\Psi(E_{10}, h, G, F_f, s) = \frac{a(2, s, k)\langle \phi_{F_f^c}(1), \phi_G(1)\rangle}{\zeta(2s+k)\zeta(4s+2k-2)\zeta(4s+2k-4)}\mathcal{L}(f, G, h^c, s)$$

where $\phi_{F_f^c}(1)$ is the first Fourier coefficient in the Fourier-Jacobi expansion of F_f^c (similarly for $\phi_G(1)$) and

$$a(2, s, k) = 2^{13-6s-6k}\pi^{6-s-2k}\,\frac{\Gamma_2(k+s-\frac{3}{2})\Gamma(s+2k-3)\Gamma(s+k-2)\Gamma(s+k-1)}{\Gamma_2(k+s)\Gamma(2k-3+2s)}$$

where

$$\Gamma_m(s) = \prod_{j=1}^{m} \Gamma(s-(j-1)/2).$$

One should note that Theorem 5.1 is stronger then the result given in [H99] where it is required that the forms have totally real Fourier coefficients. It is mentioned there that such a restriction is purely technical. If one follows through the proof of the formula without the restriction and applies Corollary 3.3 one arrives at the above result. One should also note that this makes sense, i.e., if $L(2s+2k-3, f^c) = 0$ for some s, then $L(2s+2k-3, f) = 0$ as well and so cancels the 0 out of the denominator. This follows from part (iii) of Theorem 1 of [Sh77].

Note that if the Fourier coefficients are all totally real the formula reduces to that given in [H99]. We now specialize this result to the main situation of interest, though we will use the above formula for orthogonality results.

Corollary 5.2. *Let $k > 6$ be even. Let $F_f \in \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ and $h \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be Hecke eigenforms with f having totally real Fourier coefficients and h a newform. Then we have*

$$\Psi(\mathcal{E}_{10}, h, F_f, F_f, 0) = \frac{2^{14-4k+z_k}3\pi^{6-k}[(k-2)!]^2 d_k\langle F_f, F_f\rangle\mathcal{L}(f, F_f, h^c, 0)}{(2k-3)k!L(k, f)\zeta(k)\zeta(2k-2)\zeta(2k-4)}.$$

Proof. This follows immediately from Theorem 5.1 and the relation between the inner products $\langle \Phi_{F_f}(1), \Phi_{F_f}(1)\rangle$ and $\langle F_f, F_f\rangle$ found in [KS89], namely that

$$\langle \Phi_{F_f}(1), \Phi_{F_f}(1)\rangle = \frac{2^{2k+1}\, 3\, \pi^k}{(k-1)!\, L(k,f)} \langle F_f, F_f\rangle.$$

□

It will be desirable in the next section to have such an inner product where $\mathcal{E}_{10}$ is replaced by something cuspidal in each of the variables Z_1, Z_2, Z_3. We now show how this can be done by using Poincare series. Let T be a half-integral symmetric n-rowed matrix (exactly the matrices occurring as indices in the Fourier expansion of a Siegel modular form on $\mathrm{Sp}_{2n}(\mathbb{Z})$.) If $n = 1$ these are just the positive integers. For $T > 0$, i.e., T is positive definite, set

$$c(n,k,T) = \pi^{n(n-1)/4}(4\pi)^{n(n+1)/2-nk}(\det T)^{(n+1)/2-k} \prod_{i=1}^{n} \Gamma\left(k - \frac{n+1}{2}\right).$$

Note that in the case of $n = 1$ and $m > 0$ one has

$$c(1,k,m) = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}.$$

Set

$$A_{2n} = \left\{ \begin{pmatrix} \pm 1 & s \\ 0 & \pm 1 \end{pmatrix} : s \in \mathbb{S}(\mathbb{Z}) \right\} \subset \mathrm{Sp}_{2n}(\mathbb{Z}).$$

The Tth Poincare series is defined by

$$P_{2n}^k(Z,T) = \sum_{\gamma \in A_{2n} \backslash \mathrm{Sp}_{2n}(\mathbb{Z})} j(\gamma, Z)^{-k} e^{2\pi i \operatorname{Tr}(T\gamma Z)}.$$

The case of $n = 1$ recovers the classical Poincare series. It is well known that $P_{2n}^k(Z,T)$ is a cusp form of weight k and level $\mathrm{Sp}_{2n}(\mathbb{Z})$. In fact, the Poincare series span the space of cusp forms ([Kl90], Chapter 6).

Theorem 5.3. *(*[Kl90]*, page 90) Given $F \in \mathcal{M}_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$, one has for each $T > 0$ as above*

$$\langle F(Z), P_{2n}^k(Z,T)\rangle = c(n,k,T) A_F(T).$$

We now return to the case of interest with $n_1 = 1$, $n_2 = n_3 = 2$. The map

$$(F_1, F_2, F_3) \mapsto \langle\langle\langle \mathcal{E}_{10} \circ \iota(Z_1, Z_2, Z_3), F_1(Z_1)\rangle, F_2(Z_2)\rangle, F_3(Z_3)\rangle$$

is an anti-linear map from $S_k(\mathrm{SL}_2(\mathbb{Z})) \times \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z})) \times \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ to $\mathbb{C}$. Thus it is represented by $\langle\langle\langle G_1(Z_1)G_2(Z_2)G_3(Z_3), \star\rangle, \star\rangle, \star\rangle = \prod_{i=1}^{3} \langle G_i(Z_i), \star\rangle$ for some $(G_1, G_2, G_3) \in S_k(\mathrm{SL}_2(\mathbb{Z})) \times \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z})) \times \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$. This gives us our cuspidal replacement for $\mathcal{E}_{10}$. One can see [GZ] Proposition 5.1 for a similar result.

However, as we are interested in arithmetic applications we need to make sure the Fourier coefficients of $G_1G_2G_3$ are still p-integral. Recall that we saw using Lemma 1.1 of [Sh83] that $\mathcal{E}_{10} \circ \iota$ is a sum of forms $F_i(Z_1)F_j(Z_2)F_l(Z_3)$ with $F_i \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ and $F_j, F_l \in \mathcal{M}_k(\mathrm{Sp}_4(\mathbb{Z}))$. Thus, we see the Fourier coefficients are indexed by m, T_1, T_2 where $m \geq 0$ and T_1, T_2 are half-integral symmetric 2-rowed matrices. We have that for all such $m > 0, T_1 > 0$, and $T_2 > 0$ that

$$\begin{aligned}&\langle\langle\langle \mathcal{E}_{10} \circ \iota(Z_1, Z_2, Z_3), P_2^k(Z_1, m)\rangle, P_4^k(Z_2, T_1)\rangle, P_4^k(Z_3, T_2)\rangle \\ &\quad = \langle G_1(Z_1), P_2^k(Z_1, m)\rangle \langle G_2(Z_2), P_4^k(Z_2, T_1)\rangle \langle G_3(Z_3), P_4^k(Z_3, T_2)\rangle.\end{aligned}$$

We can now apply Theorem 5.3 to conclude that for all $m > 0, T_1 > 0$, and $T_2 > 0$ the Fourier coefficients of $\mathcal{E}_{10} \circ \iota$ are the same as those of $G_1 G_2 G_3$ and hence $G_1 G_2 G_3$ has p-integral Fourier coefficients as desired.

6. An easy lemma on Saito-Kurokawa lift congruences

Let $F_f \in \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ be the Saito-Kurokawa lift of a newform $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$. In this section we show that if there exists $G \in \mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$ so that $F_f \equiv G (\operatorname{mod} \varpi)$, then there exists a Hecke eigenform $H \in \mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$ so that $F_f \equiv_{\mathrm{ev}} H (\operatorname{mod} \varpi)$. Recall that we write a congruence between two Siegel modular forms as $F \equiv G (\operatorname{mod} \varpi^m)$ to indicate $\operatorname{ord}_\varpi(A_F(T) - A_G(T)) \geq m$ for all T, i.e., that we have a congruence between the Fourier coefficients of F and G. When we wish to indicate a congruence between eigenvalues we will write $F \equiv_{\mathrm{ev}} G (\operatorname{mod} \varpi)$.

Let $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}), \mathcal{O})$ be an eigenform with $\mathcal{O}$ the ring of integers of a finite extension $E/\mathbb{Q}_p$ with uniformizer ϖ and residue field $\mathbb{F}$. We have that F gives rise to an $\mathcal{O}$-algebra homomorphism $\mathbb{T}_{\mathcal{O}}^S \to \mathcal{O}$ given by $t \mapsto \lambda_F(t)$ where $tF = \lambda_F(t) F$. We denote this map by λ_F and the composition of λ_F with the natural surjection $\mathcal{O} \to \mathbb{F}$ by $\overline{\lambda}_F$. Let $\mathfrak{m}_F$ be the kernel of $\overline{\lambda}_F$. One has that $\widetilde{F} \equiv_{\mathrm{ev}} F (\operatorname{mod} \varpi)$ if and only if $\mathfrak{m}_F = \mathfrak{m}_{\widetilde{F}}$. Moreover, one has that these maximal ideals exhaust all the maximal ideals of $\mathbb{T}_{\mathcal{O}}^S$ and

$$\mathbb{T}_{\mathcal{O}}^S = \prod_{\mathfrak{m}} \mathbb{T}_{\mathcal{O},\mathfrak{m}}^S$$

where $\mathbb{T}_{\mathcal{O},\mathfrak{m}}^S$ denotes the localization of $\mathbb{T}_{\mathcal{O}}^S$ at $\mathfrak{m}$. One has the analogous results for $\mathbb{T}_{\mathcal{O}}$ as well. From this the following lemma is immediate.

Lemma 6.1. *Let $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ be an eigenform. There exists a Hecke operator $t_F \in \mathbb{T}_{\mathcal{O}}^S$ so that $t_F F = F$ and $t_F \widetilde{F} = 0$ for all eigenforms $\widetilde{F}$ with $F \not\equiv_{\mathrm{ev}} \widetilde{F} (\operatorname{mod} \varpi)$.*

Let $G_1, \dots, G_m$ be an orthogonal Hecke eigenbasis of $\mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$. Enlarge E if necessary so that $F_f, G, G_1, \dots, G_m$ are all defined over $\mathcal{O}$. Given F_f and G as above, write $G = \sum_{i=1}^m c_i G_i$ for some $c_i \in \mathbb{C}$. We have the following theorem.

Lemma 6.2. *With the set-up as above, there exists an $i \in \{1, \dots, m\}$ so that $F_f \equiv_{\mathrm{ev}} F_i (\operatorname{mod} \varpi)$.*

Proof. We apply the Hecke operator t_{F_f} to G. Thus, we have

$$F_f \equiv t_{F_f} G (\operatorname{mod} \varpi).$$

Thus, we have that $t_{F_f} F_i$ must be nonzero modulo ϖ for at least one i with $1 \leq i \leq m$ as we know there exists a T_0 so that $A_{F_f}(T_0)$ is a ϖ-unit and so $F_f \not\equiv 0 (\bmod\, \varpi)$. Thus, we have an eigenvalue congruence between F_f and F_i for such an i, as claimed. □

7. A congruence

In this section we will produce a congruence of Fourier coefficients between a Saito-Kurokawa lift and a cuspidal eigenform $F \in \mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$. Combining this with the results in the previous section gives the eigenvalue congruence we desire.

Let $k > 6$ be even and $p > 2k-2$ a prime. Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a newform with totally real Fourier coefficients and F_f the Saito-Kurokawa lift of f. Let $f_0 = f, f_1, \ldots, f_m$ be an orthogonal Hecke eigenbasis of newforms for $S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$. Clearly we have that $F_0 = F_f, F_1 = F_{f_1}, \ldots, F_m = F_{f_m}$ is then an orthogonal Hecke eigenbasis of $\mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$. Let $F_{m+1}, \ldots, F_M$ be an orthogonal Hecke eigenbasis of $\mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$ so that $F_0, \ldots, F_M$ is an orthogonal Hecke eigenbasis of $\mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$. Observe that by Corollary 3.3 and the main result of [Ga92] on the equivariance of Petersson products under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ one has that $F_0^c, \ldots, F_m^c$ is also an orthogonal Hecke eigenbasis of $\mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ and $F_0^c, \ldots, F_M^c$ is an orthogonal eigenbasis of $\mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$. Finally let $h_0, \ldots, h_{m_1}$ an orthogonal basis of newforms of $S_k(\mathrm{SL}_2(\mathbb{Z}))$. Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ so that these bases are all defined over $\mathcal{O}$. We let ϖ denote the uniformizer of E and $\mathbb{F}$ the residue field.

Our goal in this section is to show that under certain hypotheses there exists a j with $m < j \leq M$ so that $F_f \equiv F_j (\bmod\, \varpi)$. Suppose there is no such j. We will produce a contradiction by constructing a form in $\mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$ congruent to F_f and then applying Lemma 6.2 to get a congruence to F_j for some j with $m < j \leq M$.

Write

$$G_1(Z_1)G_2(Z_2)G_3(Z_3) = \sum_{i,j,l} c_{i,j,l} h_i(Z_1) F_j^c(Z_2) F_l(Z_3) \tag{2}$$

for some $c_{i,j,l} \in \mathbb{C}$. As was done in [JB2], we wish to use the ϖ-divisibility of a multiple of $c_{0,0,0}$ to "control" a congruence. The difficulty here is that while the inner product relation of Shimura used in [JB2] allowed one to "diagonalize" the expansion of the pullback of the Eisenstein series used in that situation, the inner product in Theorem 5.1 allows us no such luxury.

We rewrite equation (2) in a more useful form for our purposes.

$$G_1(Z_1)G_2(Z_2)G_3(Z_3) = \sum_{\substack{0\leq i\leq m_1\\ 0\leq j\leq M}} c_{i,j,0}h_i(Z_1)F_j^c(Z_2)F_f(Z_3) + \sum_{\substack{0\leq i\leq m_1\\ 0\leq j\leq M\\ 0<l\leq M}} c_{i,j,l}h_i(Z_1)F_j^c(Z_2)F_l(Z_3). \tag{3}$$

The concern now is in removing as many of the forms h_i and F_j^c with $i \neq 0 \neq j$ in the first summation on the right hand side of equation (3).

Lemma 7.1. *For all $0 \leq i \leq m_1$ and all $0 \leq j, l \leq m$ with $j \neq l$ one has $c_{i,j,l} = 0$.*

Proof. Let $0 \leq j_0, l_0 \leq m$. On one hand, using the orthogonality of the eigenbases we have

$$\begin{aligned}\langle G_1, h_{i_0}\rangle\langle G_2, F_{j_0}^c\rangle\langle G_3, F_{l_0}\rangle &= \sum_{\substack{0\leq i\leq m_1\\ 0\leq j,l\leq M}} c_{i,j,l}\langle h_i, h_{i_0}\rangle\langle F_j^c, F_{j_0}^c\rangle\langle F_l, F_{l_0}\rangle \\ &= c_{i_0,j_0,l_0}\langle h_{i_0}, h_{i_0}\rangle\langle F_{j_0}^c, F_{j_0}^c\rangle\langle F_{l_0}, F_{l_0}\rangle.\end{aligned}$$

Solving this equation for c_{i_0,j_0,l_0} gives

$$c_{i_0,j_0,l_0} = \frac{\langle G_1, h_{i_0}\rangle\langle G_2, F_{j_0}^c\rangle\langle G_3, F_{l_0}\rangle}{\langle h_{i_0}, h_{i_0}\rangle\langle F_{j_0}^c, F_{j_0}^c\rangle\langle F_{l_0}, F_{l_0}\rangle}.$$

We now apply Theorem 5.1 along with the discussion where we defined G_1, G_2 and G_3 to conclude that

$$c_{i_0,j_0,l_0} = \frac{2^{z_k-1}d_k a(2,0,k)\langle \Phi_{F_{l_0}^c}(1), \Phi_{F_{j_0}^c}(1)\rangle \mathcal{L}(f_{l_0}, F_{j_0}^c, h_{i_0}^c, 0)}{\zeta(k)\zeta(2k-2)\zeta(2k-4)\langle h_{i_0}, h_{i_0}\rangle\langle F_{j_0}^c, F_{j_0}^c\rangle\langle F_{l_0}, F_{l_0}\rangle}$$

The fact that our restriction on j_0 and l_0 puts $F_{j_0}^c$ and $F_{l_0}^c$ in $\mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ allows us to use part (ii) of Theorem 2 in [KS89] to conclude that for $j_0 \neq l_0$

$$\langle \Phi_{F_{l_0}^c}(1), \Phi_{F_{j_0}^c}(1)\rangle = 0.$$

Thus we have that $c_{i_0,j_0,l_0} = 0$ for $j_0 \neq l_0$. $\square$

If there exists F_i with $m+1 \leq i \leq M$ so that $F_f \equiv_{\mathrm{ev}} F_i^c \pmod{\varpi}$ then we have a contradiction since by assumption $F_f^c = F_f$ and so we would have $F_f \equiv F_i \pmod{\varpi}$, which we assumed does not happen. Thus, we act on equation (3) by $t_{F_f} = t_{F_f^c}$ as given in Lemma 6.1 on the Z_2 variable to kill all F_i^c with $m+1 \leq i \leq M$ which, combined with Lemma 7.1 results in the equation

$$G_1(Z_1) t_{F_f} G_2(Z_2) G_3(Z_3) = \sum_{0 \le i \le m_1} c_{i,0,0} h_i(Z_1) F_f(Z_2) F_f(Z_3) + \sum_{\substack{0 \le i \le m_1 \\ 0 < j \le m}} c_{i,j,j} h_i(Z_1) t_{F_f} F_j^c(Z_2) F_j(Z_3) + \sum_{\substack{0 \le i \le m_1 \\ 0 \le j \le m \\ m < l \le M}} c_{i,j,l} h_i(Z_1) t_{F_f} F_j^c(Z_2) F_l(Z_3). \tag{4}$$

We have the following theorem that will help us remove the extra h_i's.

Theorem 7.2. *([JB2], Theorem 5.4) Let $h_{i_0} \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a newform such that the residual Galois representation $\overline{\rho}_{h_{i_0}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}})$ is irreducible and h_{i_0} is ordinary at ϖ. Then there exists a Hecke operator $t_{h_{i_0}} \in \mathbb{T}_{\mathcal{O}}$ such that*

$$t_{h_{i_0}} h_i = \begin{cases} \alpha_{h_{i_0}} h_{i_0} & \text{if } i = i_0, \\ 0 & \text{if } i \neq i_0,\ i \neq i_{m_1}, \end{cases}$$

where $\alpha_{h_{i_0}} = \frac{u_{i_0} \langle h_{i_0}, h_{i_0} \rangle}{\Omega^+_{h_{i_0}} \Omega^-_{h_{i_0}}}$ and $u_{i_0} \in \mathcal{O}^\times$.

Assume that h_0 is ordinary at ϖ and $\overline{\rho}_{h_0}$ is irreducible. Applying t_{h_0} to equation (4) gives

$$t_{h_0} G_1(Z_1) t_{F_f} G_2(Z_2) G_3(Z_3) = c_{0,0,0} \alpha_{h_0} h_0(Z_1) F_f(Z_2) F_f(Z_3) + \sum_{0 < j \le m} c_{0,j,j} \alpha_{h_0} h_0(Z_1) t_{F_f} F_j^c(Z_2) F_j(Z_3) + \sum_{\substack{0 \le j \le m \\ m < l \le M}} c_{0,j,l} \alpha_{h_0} h_0(Z_1) t_{F_f} F_j^c(Z_2) F_l(Z_3). \tag{5}$$

At this point it is possible to produce a congruence between F_f and another Siegel cusp form distinct from F_f. This congruence is "controlled" by $c_{0,0,0}\alpha_{h_0}$. However, at this point the cusp form would not necessarily be in $\mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$, so would not allow us to apply Lemma 6.2 to reach a contradiction. We need one more Hecke operator to remove the "Saito-Kurokawa" part.

Note that since the Saito-Kurokawa correspondence is Hecke-equivariant, if we assume that f is ordinary at ϖ and $\overline{\rho}_f$ is irreducible then Theorem 7.2 gives a Hecke operator $t_f^S \in \mathbb{T}_{\mathcal{O}}^S$ so that

$$t_f^S F_i = \begin{cases} \alpha_f F_f & \text{if } i = 0, \\ 0 & \text{if } 1 \le i \le m, \end{cases}$$

Set $d_{0,0,0} = \alpha_f \alpha_{h_0} c_{0,0,0}$. Assume now that f and h_0 are ordinary at ϖ and $\overline{\rho}_f$ and $\overline{\rho}_{h_0}$ are irreducible. We act on equation (5) by t_f^S to remove the extraneous Saito-Kurokawa lifts in the Z_3 variable. This allows us to rewrite equation (5) as

$$\begin{aligned} t_{h_0}G_1(Z_1)t_{F_f}G_2(Z_2)t_f^S G_3(Z_3) &= d_{0,0,0}h_0(Z_1)F_f(Z_2)F_f(Z_3) \\ &+ \sum_{\substack{0<j\leq m \\ m<l\leq M}} \alpha_{h_0}c_{0,j,l}h_0(Z_1)t_{F_f}F_j^c(Z_2)t_f^S F_l(Z_3). \end{aligned}$$

Before studying the ϖ-divisibility of $d_{0,0,0}$, we show how it "controls" a congruence. Suppose we can write $d_{0,0,0} = \mathcal{U}\varpi^{-m}$ for $\mathcal{U} \in \mathcal{O}^\times$ and $m \geq 1$. We now multiply equation (4) through by ϖ^m, take Fourier expansions in terms of Z_1 and Z_2, equate the first Fourier coefficients in the Z_1 expansion and the T_0th Fourier coefficients in the Z_2 expansion where T_0 is as given in Corollary 3.2, and look modulo ϖ. We use the fact that $G_1G_2G_3$ has ϖ-integral Fourier coefficients to obtain

$$(-\mathcal{U})F_f(Z_3) \equiv \sum_{m<l\leq M} \varpi^m \alpha_{h_0}\alpha_f c_{0,0,l} t_f^S F_l(Z_3) (\operatorname{mod} \varpi).$$

Applying Lemma 6.2 to this congruence produces a congruence $F_f \equiv_{\mathrm{ev}} t_f^S F_l (\operatorname{mod} \varpi)$ for some $m < l \leq M$. We now use that the Hecke operators respect the decomposition $\mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z})) = \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z})) \oplus \mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$ to write $t_f^S F_l = \sum_{i=m+1}^M a_i F_i$ for some $a_i \in \mathbb{C}$. One more application of Lemma 6.2 then gives the desired contradiction. In the next section we will study $d_{0,0,0}$ to give more explicit conditions on when one can write $d_{0,0,0} = \mathcal{U}\varpi^{-m}$ as above, but we conclude this section with the following theorem.

Theorem 7.3. *Let $k > 6$ be even and $p > 2k-2$ a prime. Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{R})$ be a newform and F_f the Saito-Kurokawa lift of f. Assume that f is ordinary at p and $\overline{\rho}_f$ is irreducible. Let $h_0 \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a newform such that h_0 is ordinary at p and $\overline{\rho}_{h_0}$ is irreducible. If there exists $\mathcal{U} \in \mathcal{O}^\times$ and $m \geq 1$ so that $d_{0,0,0} = \mathcal{U}\varpi^{-m}$, then there exists an eigenform $G \in \mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$ so that $F_f \equiv_{\mathrm{ev}} G (\operatorname{mod} \varpi)$.*

8. Studying the coefficient $d_{0,0,0}$

We retain the notation and assumptions from the last section for this section. We saw in the last section how $d_{0,0,0}$ can be used to "control" a congruence when one can write $d_{0,0,0} = \frac{\mathcal{U}}{\varpi^m}$ for $\mathcal{U} \in \mathcal{O}^\times$ and $m \geq 1$. In this section we determine conditions under which we can guarantee such a m and $\mathcal{U}$.

The fact that our bases were chosen to be orthogonal can be combined with equation (2) and Corollary 5.2 to conclude that

$$c_{0,0,0} = \frac{\Psi(\mathcal{E}_{10}, h_0, F_f, F_f, 0)}{\langle h_0, h_0\rangle \langle F_f, F_f\rangle^2}.$$

Thus, we have

$$\begin{aligned} d_{0,0,0} &= \alpha_{h_0}\alpha_f c_{0,0,0} \\ &= \frac{u_{h_0}u_f\Psi(\mathcal{E}_{10},h_0,F_f,F_f,0)}{\Omega^+_{h_0}\Omega^-_{h_0}\Omega^+_f\Omega^-_f\langle F_f,F_f\rangle}\frac{\langle f,f\rangle}{\langle F_f,F_f\rangle} \\ &= \frac{u_{h_0}u_f\Psi(\mathcal{E}_{10},h_0,F_f,F_f,0)}{\Omega^+_{h_0}\Omega^-_{h_0}\Omega^+_f\Omega^-_f\langle F_f,F_f\rangle}\frac{2^5 3^2\pi|D|^k L(k-1,f,\chi_D)}{|D|^{3/2}c_{g_f}(|D|)^2(k-1)L(k,f)} \\ &= \mathcal{A}\frac{d_k\pi^{7-k}L(2k-4,f)L(k-1,f,\chi_D)L(2k-3,F_f\times h_0^c)}{\zeta(k)\zeta(2k-2)\zeta(2k-4)\Omega^+_{h_0}\Omega^-_{h_0}\Omega^+_f\Omega^-_f L(k,f)^2} \end{aligned}$$

where

$$\mathcal{A} = \frac{2^{19-4k-z_k}3^3[(k-2)!]^2|D|^k}{(2k-3)(k-1)k!|D|^{3/2}c_{g_f}(|D|)^2}$$

and we have used Theorem 3.4 and Corollary 5.2. Our goal now is to determine conditions so that $\mathrm{ord}_\varpi(d_{0,0,0}) < 0$. We begin by observing that if $\gcd(p,D) = 1$ then the fact that $c_{g_f}(|D|) \in \mathcal{O}$ and $p > 2k-2$ by assumption, we must have $\mathrm{ord}_\varpi(\mathcal{A}) \leq 0$. We will need the following results in normalizing our L-values.

Theorem 8.1. *(*[Sh77]*, Theorem 1) Let $f \in S_m(\mathrm{SL}_2(\mathbb{Z}),\mathcal{O})$ be a newform. There exist complex periods $\Omega_f^\pm$ such that for each integer a with $0 < a < m$ and every Dirichlet character χ one has*

$$\frac{L(a,f,\chi)}{\tau(\chi)(2\pi i)^a} \in \begin{cases} \Omega_f^-\mathcal{O}_\chi & \text{if } \chi(-1) = (-1)^a, \\ \Omega_f^+\mathcal{O}_\chi & \text{if } \chi(-1) = (-1)^{a-1}, \end{cases}$$

where $\tau(\chi)$ is the Gauss sum of χ and $\mathcal{O}_\chi$ is the extension of $\mathcal{O}$ generated by the values of χ. We will write $L_{\mathrm{alg}}(a,f,\chi)$ to denote $\frac{L(a,f,\chi)}{\tau(\chi)(2\pi i)^a\Omega_f^\pm}$ where $\pm$ is chosen appropriately.

Proposition 8.2. *(*[JB3]*, Proposition 7.5) Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a newform, $h \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ a newform, and F_f the Saito-Kurokawa lift of f. Then one has*

$$L(s,F_f\times h) = L(s+1-k,h)L(s,f)L(s,f\times h)$$

where $L(s,f\times h)$ is the Rankin convolution L-function as defined in the introduction.

Theorem 8.3. *(*[JB3]*, Theorem 7.3) Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ and $h \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a newforms. If k is even then*

$$L_{\mathrm{alg}}(2k-3,f\times h) := \frac{L(2k-3,f\times h)}{\pi^{2k-3}\Omega_f^+\Omega_h^-} \in \overline{\mathbb{Q}}.$$

In order to apply the previous theorem, we now assume that our h_0 has totally real Fourier coefficients. We also make use of the well known result that $\zeta_{\mathrm{alg}}(a) := \frac{\zeta(a)}{\pi^k} \in \mathbb{Q}$. Combining all of these we can write

$$d_{0,0,0} = \mathcal{A}\frac{d_k(2\pi i)^{3k-10}\mathscr{L}(k,f,h_0)}{\zeta_{\mathrm{alg}}(k)\zeta_{\mathrm{alg}}(2k-2)\zeta_{\mathrm{alg}}(2k-4)L_{\mathrm{alg}}(k,f)^2}$$

where

$$\mathscr{L}(k,f,h_0) = L_{\mathrm{alg}}(2k-3,f)L_{\mathrm{alg}}(2k-4,f)L_{\mathrm{alg}}(k-1,f,\chi_D)L_{\mathrm{alg}}(k-2,h_0)L_{\mathrm{alg}}(2k-3,f\times h_0).$$

Thus, we see that $\mathrm{ord}_\varpi(d_{0,0,0}) < 0$ if we have

(6)
$$-m = \mathrm{ord}_\varpi(\zeta_{\mathrm{alg}}(k)\zeta_{\mathrm{alg}}(2k-2)\zeta_{\mathrm{alg}}(2k-4)L_{\mathrm{alg}}(k,f)^2) - \mathrm{ord}_\varpi\left(d_k\mathscr{L}(k,f,h_0)\right) > 0.$$

We summarize with the following theorem.

Theorem 8.4. *Let $k > 6$ be even and $p > 2k-2$ a prime. Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}),\mathbb{R})$ be a newform and F_f the Saito-Kurokawa lift of f. Assume that f is ordinary at p and $\overline{\rho}_f$ is irreducible. Let $h_0 \in S_k(\mathrm{SL}_2(\mathbb{Z}),\mathbb{R})$ be a newform such that h_0 is ordinary at p and $\overline{\rho}_{h_0}$ is irreducible. If*

$$-m = \mathrm{ord}_\varpi(\zeta_{\mathrm{alg}}(k)\zeta_{\mathrm{alg}}(2k-2)\zeta_{\mathrm{alg}}(2k-4)L_{\mathrm{alg}}(k,f)^2) - \mathrm{ord}_\varpi\left(d_k\mathscr{L}(k,f,h_0)\right) > 0$$

then there exists an eigenform $G \in \mathcal{S}_k^{\mathrm{NM}}(\mathrm{Sp}_4(\mathbb{Z}))$ so that $F_f \equiv_{\mathrm{ev}} G \pmod{\varpi}$.

9. Selmer groups

Let K be a field and M a topological $\mathrm{Gal}(\overline{K}/K)$-module. We write the cohomology group $\mathrm{H}^1_{\mathrm{cont}}(\mathrm{Gal}(\overline{K}/K), M)$ as $\mathrm{H}^1(K,M)$ where "cont" refers to continuous cocycles. For a prime ℓ, we write D_ℓ for the decomposition group at ℓ and identify it with $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$.

Let $E/\mathbb{Q}_p$ be a finite extension. Let $\mathcal{O}$ be the ring of integers of E and ϖ a uniformizer. Let V be a finite dimensional Galois representation over E. We will also find it convenient to write $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(E)$ to denote the Galois representation V when $\dim_E(V) = n$. We switch interchangably between these notations depending upon context. Let $T \subseteq V$ be a Galois-stable $\mathcal{O}$-lattice, i.e., T is stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $T \otimes_{\mathcal{O}} E \cong V$. Set $W = V/T$.

We write $\mathbb{B}_{\mathrm{crys}}$ for the ring of p-adic periods ([Fo82]). Set

$$\mathrm{Crys}(V) = \mathrm{H}^0(\mathbb{Q}_p, V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{crys}}).$$

We say the representation V is *crystalline* if $\dim_{\mathbb{Q}_p} V = \dim_{\mathbb{Q}_p} \mathrm{Crys}(V)$. Let $\mathrm{Fil}^i \mathrm{Crys}(V)$ be a decreasing filtration of $\mathrm{Crys}(V)$. If V is crystalline, we say V is *short* if $\mathrm{Fil}^0 \mathrm{Crys}(V) = \mathrm{Crys}(V)$, $\mathrm{Fil}^p \mathrm{Crys}(V) = 0$, and if whenever V' is a nonzero quotient of V, then $V' \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(p-1)$ is ramified. Note that $\mathbb{Q}_p(n)$ is the 1-dimensional space over $\mathbb{Q}_p$ on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via the nth power of the p-adic cyclotomic character.

The local Selmer groups are defined as follows. Set

$$\mathrm{H}^1_f(\mathbb{Q}_\ell, V) = \begin{cases} \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, V) & \ell \neq p \\ \ker(\mathrm{H}^1(\mathbb{Q}_p, V) \to \mathrm{H}^1(\mathbb{Q}_p, V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{crys}})) & \ell = p \end{cases}$$

where

$$\mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, M) = \ker(\mathrm{H}^1(\mathbb{Q}_\ell, M) \to \mathrm{H}^1(I_\ell, M))$$

for any D_ℓ-module M where I_ℓ is the inertia group at ℓ. With W as above we define $\mathrm{H}^1_f(\mathbb{Q}_\ell, W)$ to be the image of $\mathrm{H}^1_f(\mathbb{Q}_\ell, V)$ under the natural map $\mathrm{H}^1(\mathbb{Q}_\ell, V) \to \mathrm{H}^1(\mathbb{Q}_\ell, W)$.

Definition 9.1. The Selmer group of W is given by

$$\mathrm{Sel}(\mathbb{Q}, W) = \ker\left(\mathrm{H}^1(\mathbb{Q}, W) \to \bigoplus_\ell \frac{\mathrm{H}^1(\mathbb{Q}_\ell, W)}{\mathrm{H}^1_f(\mathbb{Q}_\ell, W)}\right),$$

i.e., it is the cocycles $\mathfrak{c} \in \mathrm{H}^1(\mathbb{Q}, W)$ that lie in $\mathrm{H}^1_f(\mathbb{Q}_\ell, W)$ when restricted to D_ℓ.

Before we proceed further with our study of Selmer groups, it is necessary to recall the relationship between extensions of modules and the first cohomology group. Let R be a commutative ring with identity and G a group. Let M and N be $R[G]$-modules that are free of finite rank as R-modules. We denote the action of G on M by ρ_M and the action of G on N by ρ_N. Recall that as M and N are $R[G]$-modules, we have an action of G on $\mathrm{Hom}_R(M, N)$ as well. The action of G on $\mathrm{Hom}_R(M, N)$ is given as follows. Let $\phi \in \mathrm{Hom}_R(M, N)$, $m \in M$ and $g \in G$, then $g \cdot \phi(m) = \rho_N(g)\phi(\rho_M(g^{-1})m)$.

An extension of M by N is a short exact sequence

$$0 \longrightarrow N \xrightarrow{\alpha} X \xrightarrow{\beta} M \longrightarrow 0$$

where X is a $R[G]$-module and α and β are $R[G]$-homomorphisms. We sometimes refer to such an extension as the extension X. We say two extensions X and Y are equivalent if there is a $R[G]$-isomorphism γ making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\alpha_X} & X & \xrightarrow{\beta_X} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ 0 & \longrightarrow & N & \xrightarrow{\alpha_Y} & Y & \xrightarrow{\beta_Y} & M & \longrightarrow & 0. \end{array}$$

Let $\mathrm{Ext}^1_{R[G]}(M, N)$ denote the set of equivalence classes of $R[G]$-extensions of M by N which split as extensions of R-modules, i.e., if X is the extension of M by N, then $X \cong M \oplus N$ as R-modules.

The following result will allow us to appropriately define the degree n Selmer group. The case where $M = N$ is given as Proposition 4 in [W95]. The proof given here is an adaptation of the proof given there.

Theorem 9.2. *Let M and N be $R[G]$-modules with G actions given by ρ_M and ρ_N respectively. There is a one-one correspondence between the sets $\mathrm{H}^1(G, \mathrm{Hom}_R(M, N))$ and $\mathrm{Ext}^1_{R[G]}(M, N)$.*

Proof. We will define a bijection from $\mathrm{Ext}^1_{R[G]}(M, N)$ to $\mathrm{H}^1(G, \mathrm{Hom}_R(M, N))$. Let

$$0 \longrightarrow N \xrightarrow{\alpha} X \xrightarrow{\beta} M \longrightarrow 0$$

be an extension of M by N. We denote the G-action on X by ρ. Let $s_X : M \to X$ be a R-section of X, i.e., s_X is a R-module homomorphism so that $\beta \circ s_X = \mathrm{id}_M$. Observe that for all $g \in G$ and all $m \in M$ we have

$$\begin{aligned} \beta(\rho(g)s_X(\rho_M(g^{-1})m) - s_X(m)) &= \rho_M(g)\beta(s_X(\rho_M(g^{-1})m)) - \beta(s_X(m)) \\ &= \rho_M(g)\rho_M(g^{-1})m - m \\ &= 0 \end{aligned}$$

where we have used that β is a $R[G]$-module and $\beta \circ s_X = \mathrm{id}_M$. Thus we have that $\rho(g)s_X(\rho_M(g^{-1}m)) - s_X(m) \in \ker(\beta)$ for all $g \in G, m \in M$, i.e., $\rho(g)s_X(\rho_M(g^{-1}m)) - s_X(m) \in \alpha(N)$ for all $g \in G, m \in M$. For $g \in G$, define $\mathfrak{c}_g : M \to N$ by

$$\mathfrak{c}_g(m) = \alpha^{-1}(\rho(g)s_X(\rho_M(g^{-1})m) - s_X(m)).$$

Our work above shows this is well-defined and it is immediate that for each $g \in G$ we have $\mathfrak{c}_g \in \mathrm{Hom}_R(M, N)$. We can show that $g \mapsto \mathfrak{c}_g$ is in $\mathrm{H}^1(G, \mathrm{Hom}_R(M, N))$. To see this we observe that

$$\begin{aligned} g_1 \cdot \mathfrak{c}_{g_2}(m) + \mathfrak{c}_{g_1}(m) &= \rho_N(g_1)\mathfrak{c}_{g_2}(\rho_M(g_1^{-1}m)) + \mathfrak{c}_{g_1}(m) \\ &= \rho_N(g_1)\left(\alpha^{-1}(\rho(g_2)s_X(\rho_M(g_2^{-1})\rho_M(g_1^{-1})m) - s_X(\rho_M(g_1^{-1}m)))\right) \\ &\quad + \alpha^{-1}(\rho(g_1)s_X(\rho_M(g_1^{-1})m) - s_X(m)) \\ &= \alpha^{-1}(\rho(g_1g_2)s_X(\rho_M((g_1g_2)^{-1})m)) - \alpha^{-1}(\rho(g_1)s_X(\rho_M(g_1^{-1})m)) \\ &\quad + \alpha^{-1}(\rho(g_1)s_X(\rho_M(g_1^{-1})m)) - \alpha^{-1}(s_X(m)) \\ &= \mathfrak{c}_{g_1g_2}(m). \end{aligned}$$

This gives a map from $\mathrm{Ext}^1_{R[G]}(M, N)$ to $\mathrm{H}^1(G, \mathrm{Hom}_R(M, N))$. We need to show that this map is well-defined. Let

$$0 \longrightarrow N \longrightarrow Y \longrightarrow M \longrightarrow 0$$

be an equivalent extension and let s_Y be a R-section of Y. Let $\gamma : X \to Y$ be the $R[G]$-isomorphism giving the equivalence of extensions. If we set

$\psi = \alpha^{-1}\gamma^{-1}(s_Y - \gamma s_X)$, then an elementary calculation shows that if $\mathfrak{c}_X$ and $\mathfrak{c}_Y$ are the cocyles arising from the extensions as above, then $(\mathfrak{c}_X)_g - (\mathfrak{c}_Y)_g = g \cdot \psi - \psi$. Thus, two equivalent extensions give rise to cocycles that differ by a coboundary, and hence give the same element of $\mathrm{H}^1(G, \mathrm{Hom}_R(M,N))$ and so the map is well-defined. It remains to show that this map is bijective.

We begin by showing our map is injective. Let

$$0 \longrightarrow N \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} M \longrightarrow 0$$

and

$$0 \longrightarrow N \xrightarrow{\alpha_Y} Y \xrightarrow{\beta_Y} M \longrightarrow 0$$

be two extensions that give rise to equivalent cocycles $\mathfrak{c}_X$ and $\mathfrak{c}_Y$, i.e., there exists $\psi \in \mathrm{Hom}_R(M,N)$ so that $(\mathfrak{c}_X)_g - (\mathfrak{c}_Y)_g = g \cdot \psi - \psi$ for all $g \in G$. Denote the G-action on X by ρ_X and on Y by ρ_Y. Let s_X be a R-section of X and s_Y be an R-section of Y. The condition on the cocycles $\mathfrak{c}_X$ and $\mathfrak{c}_Y$ can be used to show that we have

$$\begin{aligned}\psi(\rho_M(g)m) = \rho_N(g)\psi(m) - \alpha_Y^{-1}(\rho_2(g)s_Y(m) - s_Y(\rho_M(g)m))\\ + \alpha_X^{-1}(\rho_X(g)s_X(m) - s_X(\rho_M(g)m)).\end{aligned}$$

The fact that $X \cong M \oplus N$ as R-modules allows us to conclude that for each $x \in X$ there exists unique $m \in M$, $n \in N$ so that $x = \alpha_X(n) + s_X(m)$. Define $\gamma : X \to Y$ by

$$\gamma(x) = \alpha_Y(n) + s_Y(m) - \alpha_Y(\psi(m)).$$

The formula given about for $\psi(\rho_M(g)m)$ allows one to show that γ is in fact an $R[G]$-module homomorphism. It is easy to see from our definition of γ that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\alpha_X} & X & \xrightarrow{\beta_X} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ 0 & \longrightarrow & N & \xrightarrow{\alpha_Y} & Y & \xrightarrow{\beta_Y} & M & \longrightarrow & 0 \end{array}$$

commutes. The Snake lemma gives that γ is an isomorphism and hence the extensions are equivalent and our map $\mathrm{Ext}^1_{R[G]}(M,N) \to \mathrm{H}^1(G, \mathrm{Hom}_R(M,N))$ is an injection.

Let $\mathfrak{c} \in \mathrm{H}^1(G, \mathrm{Hom}_R(M,N))$. Define $X = M \oplus N$ as an R-module. We now show that we can define a G-action on X so that the resulting extension gives rise to $\mathfrak{c}'$ under the mapping $\mathrm{Ext}^1_{R[G]}(M,N) \to \mathrm{H}^1(G, \mathrm{Hom}_R(M,N))$. Define the G-action on X via

$$\rho(g)(m,n) = (\rho_M(g)m, \mathfrak{c}_g(\rho_M(g)m))$$

where we use the notation $\mathfrak{c}_g \in \mathrm{Hom}_R(M,N)$ as above. If we write $\mathfrak{c}_X$ for the cocycle arising from the extension X, a short calculation gives that $\mathfrak{c} = \mathfrak{c}_X$. Thus, the map $\mathrm{Ext}^1_{R[G]}(M,N) \to \mathrm{H}^1(G, \mathrm{Hom}_R(M,N))$ is a surjection and hence from what we have already shown a bijection as claimed. $\square$

Let $W[n]$ be the $\mathcal{O}$-submodule of W consisting of elements killed by ϖ^n. The previous theorem gives a bijection between $\mathrm{Ext}^1_{(\mathcal{O}/\varpi^n)[D_p]}(\mathcal{O}/\varpi^n, W[n])$ and $\mathrm{H}^1(D_p, W[n])$. For $\ell \neq p$, we define the local degree n Selmer groups by $\mathrm{H}^1_f(\mathbb{Q}_\ell, W[n]) = \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}, W[n])$. At the prime p we define the local degree n Selmer group to be the subset of classes of extensions of D_p-modules

$$0 \longrightarrow W[n] \longrightarrow X \longrightarrow \mathcal{O}/\varpi^n \longrightarrow 0$$

where X lies in the essential image of the functor $\mathbb{V}$ defined in § 1.1 of [DFG04]. The precise definition of $\mathbb{V}$ is technical and is not needed here. We content ourselves with stating that this essential image is stable under direct sums, subobjects, and quotients ([DFG04], § 2.1). For our purposes the following two propositions are what is needed.

Proposition 9.3. *([DFG04], p. 670) If V is a short crystalline representation at p, T a D_p-stable lattice, and X a subquotient of T/ϖ^n that gives an extension of D_p-modules as above, then the class of this extension is in $\mathrm{H}^1_f(\mathbb{Q}_p, W[n])$.*

Proposition 9.4. *([JB2], Proposition 7.9) Let $\mathfrak{c}$ be a non-zero cocycle in $\mathrm{H}^1(\mathbb{Q}, W[1])$ and assume that T/ϖ is irreducible. If $\mathfrak{c}|_{D_\ell} \in \mathrm{H}^1_f(\mathbb{Q}_\ell, W[1])$ is non-zero, then $\mathfrak{c}|_{D_\ell}$ gives a non-zero ϖ-torsion element of $\mathrm{H}^1_f(\mathbb{Q}_\ell, W)$. Moreover, if $\mathfrak{c}|_{D_\ell} \in \mathrm{H}^1_f(\mathbb{Q}_\ell, W[1])$ for every prime ℓ, then $\mathfrak{c}$ is a non-zero ϖ-torsion element of $\mathrm{Sel}(\mathbb{Q}, W)$.*

10. Galois representations and Selmer groups

In this section we show how given a congruence as in Theorem 8.4 one has that $\mathrm{Sel}(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k)) \neq 0$ and $p \mid \#\,\mathrm{Sel}(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$. We will mainly summarize results found in ([JB2], section 8) so the interested reader is advised to consult there for the details.

Let $E/\mathbb{Q}_p$ be a finite extension as before large enough so that our results from section 6 are defined over E. We enlarge E when necessary so that the appropriate Galois representations in this section are defined over E as well. Let $\mathcal{O}$ be the ring of integers of E, ϖ the uniformizer, $\mathfrak{p} = (\varpi)$ the prime ideal over p, and $\mathbb{F}$ the residue field.

Let $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(V_{f,\mathfrak{p}})$ be the p-adic Galois representation associated to an eigenform f, $T_{f,\mathfrak{p}}$ a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathcal{O}$-lattice, and $W_{f,\mathfrak{p}} = V_{f,\mathfrak{p}}/T_{f,\mathfrak{p}}$. We denote twists by the mth power of the cyclotomic character by writing $V_{f,\mathfrak{p}}(m)$ and similarly for $W_{f,\mathfrak{p}}(m)$. We also have the following result giving the existence of 4-dimensional Galois representations attached to Siegel eigenforms.

Theorem 10.1. *([SU06], Theorem 3.1.3) Let $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ be an eigenform, K_F the number field generated by the Hecke eigenvalues of F, and $\wp$ a prime of K_F over p. There exists a finite extension E of the completion of $K_{F,\wp}$ of K_F at $\wp$ and a continuous semi-simple Galois representation*

$$\rho_{F,\wp} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(E)$$

unramified away from p so that for all $\ell \neq p$ we have

$$\det(X \cdot 1_4 - \rho_{F,\wp}(\mathrm{Frob}_\ell)) = L_{\mathrm{spin},(\ell)}(X).$$

The following result is crucial in producing elements in the Selmer group.

Theorem 10.2. *([F89], [U05]) Let F be as in Theorem 10.1. The restriction of $\rho_{F,\wp}$ to the decomposition group D_p is crystalline at p. In addition, if $p > 2k-2$ then $\rho_{F,\wp}$ is short.*

Suppose that we have an eigenvalue congruence $F_f \equiv_{\mathrm{ev}} G(\mathrm{mod}\, \varpi)$ as in Theorem 8.4. This congruence combined with Proposition 3.5, Theorem 10.1, and the Brauer-Nesbitt Theorem give that $\overline{\rho}_{G,\mathfrak{p}}^{\mathrm{ss}} = \omega^{k-1} \oplus \overline{\rho}_{f,\mathfrak{p}} \oplus \omega^{k-2}$. The goal is to study $\overline{\rho}_{G,\mathfrak{p}}$ and use this Galois representation to produce a nontrivial p-torsion element in $\mathrm{Sel}(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$. One can use linear algebra along with the fact that $\rho_{G,\mathfrak{p}}$ is irreducible to deduce the following proposition.

Proposition 10.3. *([JB2], p. 316) There is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable lattice $T_{G,\mathfrak{p}}$ so that the reduction $\overline{\rho}_{G,\mathfrak{p}}$ is of the form*

$$\overline{\rho}_{G,\mathfrak{p}} = \begin{pmatrix} \omega^{k-2} & *_1 & *_2 \\ *_3 & \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ 0 & 0 & \omega^{k-1} \end{pmatrix}$$

where $*_1$ or $*_3$ is zero and so that $\overline{\rho}_{G,\mathfrak{p}}$ is not equivalent to a representation with $*_2$ and $*_4$ both zero.

We begin by assuming that $*_3 = 0$. We would like to show that the quotient

$$\begin{pmatrix} \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ 0 & \omega^{k-1} \end{pmatrix}$$

is not split. If we can show this, then we can twist $\overline{\rho}_{G,\mathfrak{p}}$ by ω^{1-k} so that $*_4$ gives a nontrivial element of $\mathrm{H}^1(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k)[1])$. Suppose it is split. In this case Proposition 10.3 gives that the quotient

$$\begin{pmatrix} \omega^{k-2} & *_2 \\ 0 & \omega^{k-1} \end{pmatrix}$$

cannot be split as well. However, in ([JB2], section 8) it is shown that this quotient being nonsplit gives a nontrivial quotient of the ω^{-1}-isotypical piece of the p-part of the class group of $\mathbb{Q}(\mu_p)$, which by Herbrand's theorem does not exist. Thus, if $*_3 = 0$ we obtain a nontrivial torsion element of $\mathrm{H}^1(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k)[1])$. In this case it remains to show that the local conditions are satisfied so that we obtain an element in the Selmer group.

Suppose now that $*_1 = 0$. In this case we obtain that

$$\begin{pmatrix} \omega^{k-2} & *_2 \\ 0 & \omega^{k-1} \end{pmatrix}$$

is a quotient extension. However, as stated above we must have $*_2 = 0$. This gives that

$$\begin{pmatrix} \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ 0 & \omega^{k-1} \end{pmatrix}$$

is a nontrivial quotient extension and we again obtain a nontrivial element of $\mathrm{H}^1(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k)[1])$ after twisting by ω^{1-k}.

It is now easy to see that we have a nontrivial torsion element of $\mathrm{Sel}(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$ by using the fact that $\rho_{G,\mathfrak{p}}$ is unramified away from p and Proposition 9.3 to see that our cocycle $\mathfrak{c} := *_4$ satisfies all of the local conditions. Thus, we have the following theorem.

Theorem 10.4. *Let $k > 6$ be even and $p > 2k-2$ a prime. Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{R})$ be a newform and F_f the Saito-Kurokawa lift of f. Assume that f is ordinary at p and $\overline{\rho}_f$ is irreducible. Let $h_0 \in S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{R})$ be a newform such that h_0 is ordinary at p and $\overline{\rho}_{h_0}$ is irreducible. If*

$$-m = \mathrm{ord}_{\varpi}(\zeta_{\mathrm{alg}}(k)\zeta_{\mathrm{alg}}(2k-2)\zeta_{\mathrm{alg}}(2k-4)L_{\mathrm{alg}}(k,f)^2) - \mathrm{ord}_{\varpi}(d_k \mathscr{L}(k,f,h_0)) > 0$$

then $\mathrm{Sel}(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k)) \neq 0$ *and* $p \mid \# \mathrm{Sel}(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$.

11. The Bloch-Kato Conjecture

In this section we recall the statement of the Bloch-Kato conjecture in our situation, observing how Theorem 10.4 gives evidence for the validity of this conjecture. We follow the excellent account given in [D08] for our exposition of the Bloch-Kato conjecture for modular forms.

Let f be a newform of weight $2k-2$ and level $\mathrm{SL}_2(\mathbb{Z})$. As above, for each prime ℓ, let $V_\ell := V_{f,\lambda}$ be the 2-dimensional ℓ-adic Galois representation associated to f. We let $T_\ell := T_{f,\lambda}$ be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable lattice and set $W_\ell := W_{f,\lambda} = V_\ell/T_\ell$. For any integer j we have a natural map $\pi : V_\ell(j) \to W_\ell(j)$ which induces a natural map of cohomology $\pi_* : \mathrm{H}^1(\mathbb{Q}, V_\ell(j)) \to \mathrm{H}^1(\mathbb{Q}, W_\ell(j))$. The Shafarevich-Tate group is defined to be

$$Ш(j) = \bigoplus_{\ell} \mathrm{H}^1_f(\mathbb{Q}, W_\ell(j))/\pi_* \mathrm{H}^1_f(\mathbb{Q}, V_\ell(j)).$$

We define the set $\Gamma_{\mathbb{Q}}(j)$ as the sum over the global sections:

$$\Gamma_{\mathbb{Q}}(j) = \bigoplus_{\ell} \mathrm{H}^0(\mathbb{Q}, W_\ell(j)).$$

The Bloch-Kato conjecture can now be stated as follows.

Conjecture 11.1. (Bloch-Kato) With the notation as above, one has

$$L_{\mathrm{alg}}(k, f) = \frac{\prod_\ell c_\ell(k)}{\#\Gamma_{\mathbb{Q}}(3-k)\#\Gamma_{\mathbb{Q}}(1-k)} \# Ш(1-k)$$

where $c_\ell(j)$ are the "Tamagawa factors".

We would like to thank Neil Dummigan for pointing out the incorrect twists given on the global sections in [JB2]. The correct values here are due to his correction of our previous mistake.

In order to see how Theorem 10.4 gives evidence for this conjecture, we make the following observations. The fact that we are assuming that $\overline{\rho}_f$ is irreducible, gives that the terms $\#\Gamma_{\mathbb{Q}}(3-k)$ and $\#\Gamma_{\mathbb{Q}}(1-k)$ both must be p-units. See ([D08], Proposition 4.1) for example.

By work of Kato ([K04], Theorem 14.2), we know that away from the central critical point the Selmer group is finite. Thus, in our case we can identify the p-part of the Selmer group with the p-part of the Shafarevich-Tate group.

It only remains to deal with the Tamagawa factors. For $\ell \neq p$, we have that $\mathrm{ord}_\varpi(c_\ell(j))$ is defined to be

$$\mathrm{length}(\mathrm{H}^0(\mathbb{Q}_\ell, W_p(j))/\, \mathrm{H}^0(\mathbb{Q}_\ell, V_p(j)^{I_p}/T_p(j)^{I_p}))$$

where I_p is the inertia group. However, as $V_p(j)$ is unramified at all $\ell \neq p$, we have that $\mathrm{ord}_\varpi(c_\ell(j)) = 0$ for all $\ell \neq p$. Thus, it only remains to handle the case of $c_p(j)$. If we further assume that $p > 3k-3$, then Theorem 4.1 (iii) of [BK90] gives that $\mathrm{ord}_\varpi(c_p(j)) = 0$ as well. Combining all of these facts shows that Theorem 10.4 when one adds the condition that $p > 3k-3$ provides evidence for the Bloch-Kato conjecture for modular forms.

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