THE FIRST NEGATIVE HECKE EIGENVALUE OF GENUS 2 SIEGEL CUSPFORMS WITH LEVEL $N \geq 1$

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Abstract. In this short note we extend results of Kohnen and Sengupta on the sign of eigenvalues of Siegel cuspforms. We show that their bound for the first negative Hecke eigenvalue of a genus 2 Siegel cuspform of level 1 extends to the case of level $N > 1$. We also discuss the signs of Hecke eigenvalues of CAP forms.

1. Introduction

Let $\mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ be the space of Siegel cuspforms of weight k and level $\mathrm{Sp}_4(\mathbb{Z}) \subset \mathrm{GL}_4(\mathbb{Z})$. Denote the space of Maass spezialchars by $\mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z})) \subset \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$. Let $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a nonzero Hecke eigenform of all Hecke operators $T(n)$ with $n > 0$. Write $\lambda_F(n)$ for the eigenvalue of $T(n)$ acting on F. It was shown by Breulmann in [B99] that $F \in \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ if and only if $\lambda_F(n) > 0$ for all $n > 0$. Essentially this boiled down to an elementary calculation combined with the fact that $F \in \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ if and only if the Spinor L-function $L(s, F, \mathrm{Spin})$ has a pole at $s = k$ ([E81]). This result naturally leads one to ask the question of what can be said about the signs of $\lambda_F(n)$ for $F \notin \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$. In [K07] it is shown that for such F the values $\lambda_F(n)$ change sign infinitely often. Furthermore, in [KS07] it is shown that if k is odd or $F \notin \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ then there exists $n \ll k^2 \log^{20} k$ such that $\lambda_F(n) < 0$ where the implied constant is absolute and effectively computable. It is then natural to ask what can be said in the case of level $\Gamma_0^2(N)$ for $N > 1$.

The natural generalization of the Maass spezialchars to the case of level $\Gamma_0^2(N)$ is the notion of CAP forms (see § 3.) We show (essentially a result of [PS08]) that Breulmann's result generalizes to the level $\Gamma_0^2(N)$ situation as well. See Theorem 3.1 for the precise result. Once we have dealt with CAP forms, we look at the case of non-CAP forms. We then generalize Kohnen and Sengupta's arguments from [KS07] to the case of level $\Gamma_0^2(N)$ to show that their bound of $n \ll k^2 \log^{20} k$ holds in this case as well. See Theorem 4.1 for the precise statement.

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2. Notation and Set-up

Throughout the paper we write $A \ll B$ to mean there is an absolute constant c so that $A \leq cB$. If the constant is not absolute, say it depends on k, we write $A \ll_k B$.

Let $\mathrm{G} = \mathrm{GSp}_4$, i.e.,

$$\mathrm{G} = \{g \in \mathrm{GL}_4 : \ {}^t g J g = \lambda(g) J, \, \lambda(g) \in \mathrm{GL}_1\}$$

where $J = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$. We have a natural map $\lambda : G \to \mathrm{GL}_1$. The kernel of this map is the familiar group Sp_4. For N a positive integer, set $\Gamma_0^2(N) \subset \mathrm{Sp}_4(\mathbb{Z})$ to be the subgroup defined by

$$\Gamma_0^2(N) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) : C \equiv 0 (\mathrm{mod}\, N) \right\}.$$

The group

$$\mathrm{G}^+(\mathbb{R}) = \{g \in \mathrm{G}(\mathbb{R}) : \lambda(g) > 0\}.$$

acts on the Siegel upper half-space

$$\mathfrak{h}^2 = \{Z \in \mathrm{M}_2(\mathbb{C}) : \ {}^t Z = Z, \, \mathrm{Im}(Z) > 0\}$$

by linear fractional transformation in the usual way.

Let $F : \mathfrak{h}^2 \to \mathbb{C}$ be a holomorphic function. The group $\mathrm{G}^+(\mathbb{R})$ acts on F via the slash operator

$$(F|_k g)(Z) = \lambda(g)^k j(g, Z)^{-k} F(gZ)$$

where $j(g, Z) = \det(CZ + D)$ is the usual automorphy factor. The space of Siegel modular forms of weight k and level $\Gamma_0^2(N)$ is the space of such F with the condition that $(F|_k g)(Z) = F(Z)$ for all $g \in \Gamma_0^2(N)$. This space is denoted $\mathcal{M}_k(\Gamma_0^2(N))$ and we denote the subspace of cusp forms by $\mathcal{S}_k(\Gamma_0^2(N))$. We have the usual Hecke operators $T(n)$ for $p \nmid N$ as defined in [A74]. We denote the Frobenius operators of Andrianov for $p \mid N$ by $T(p)$, see [A01] for example for the definition.

Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a Hecke eigenform. Associated to F is a cuspidal automorphic form Φ_F defined as follows. Write $N = \prod p^{r_p}$ (we set $r_p = 0$ for $p \nmid N$) and define $\mathrm{K}_0(N)$ by

$$\mathrm{K}_0(N) = \prod_{p \nmid \infty} \mathrm{K}_0(p^{r_p})$$

where

$$\mathrm{K}_0(p^{r_p}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{G}(\mathbb{Z}_p) : C \equiv 0 (\mathrm{mod}\, p^{r_p} \mathbb{Z}_p) \right\}.$$

Strong approximation for $\mathrm{G}(\mathbb{A})$ allows us to write

$$\mathrm{G}(\mathbb{A}) = \mathrm{G}(\mathbb{Q})\, \mathrm{G}^+(\mathbb{R})\, \mathrm{K}_0(N).$$

Thus, given $g \in \mathrm{G}(\mathbb{A})$ there exists $g_{\mathbb{Q}} \in \mathrm{G}(\mathbb{Q})$, $g_\infty \in \mathrm{G}^+(\mathbb{R})$, and $k_0 \in \mathrm{K}_0(N)$ such that $g = g_{\mathbb{Q}} g_\infty k_0$. Define $\Phi_F : \mathrm{G}(\mathbb{A}) \to \mathbb{C}$ by

$$\Phi_F(g) = (F \mid g_\infty)(i1_2).$$

Note that Φ_F is well defined since F has level $\Gamma_0^2(N)$ and

$$\Gamma_0^2(N) = \mathrm{G}(\mathbb{Q}) \cap \mathrm{G}^+(\mathbb{R})\,\mathrm{K}_0(N).$$

Let V_F denote the space of right translates of Φ_F. The group $\mathrm{G}(\mathbb{A})$ acts on V_F by right translation. We can decompose the space V_F into a finite direct sum of irreducible cuspidal automorphic representations of $\mathrm{G}(\mathbb{A})$. Let π_F be one of these irreducible components and write $\pi_F = \otimes \pi_{F,p}$.

Let χ_1, χ_2 and σ be unramified characters of $\mathbb{Q}_p^\times$. Denote by $\chi_1 \times \chi_2 \rtimes \sigma$ the representation of $\mathrm{G}(\mathbb{Q}_p)$ induced from the character of the Borel subgroup of $\mathrm{G}(\mathbb{Q}_p)$ given by

$$\begin{pmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & b_1 a_1^{-1} & \\ & & * & b_1 a_2^{-1} \end{pmatrix} \mapsto \chi_1(a_1)\chi_2(a_2)\sigma(b_1).$$

Adopting the notation of [ST], for $p \nmid N$ we have that $\pi_{F,p}$ is isomorphic to the Langlands quotient of an induced representation of the form $\chi_1 \times \chi_2 \rtimes \sigma$.

We can attach a degree 4 Spinor L-function to F as either the Langlands Spinor L-function or the Andrianov Spinor L-function. The only difference is at the primes $p \mid N$. When we wish to refer to the Langlands L-function we will always use the notation $L(s, \pi_F, \mathrm{Spin})$ and for the Andrianov L-function we will write $L(s, F, \mathrm{Spin})$. For $p \nmid N$, the pth Euler factor is given by

$$\begin{aligned} L_p(s, \pi_{F,p}, \mathrm{Spin}) &= L_p(s, F, \mathrm{Spin}) \\ &= [(1-\alpha_{p,0}p^{-s})(1-\alpha_{p,0}\alpha_{p,1}p^{-s})(1-\alpha_{p,0}\alpha_{p,2}p^{-s})(1-\alpha_{p,0}\alpha_{p,1}\alpha_{p,2}p^{-s})]^{-1} \end{aligned}$$

where $\alpha_{p,1} = \chi_1(p)$, $\alpha_{p,2} = \chi_2(p)$, and $\alpha_{p,0} = \sigma(p)$ are the pth Satake parameters of F. We have normalized the L-function here is a somewhat non-standard manner. It amounts to substituting $s+k-3/2$ for s in Andrianov's normalization. Note that by our choice of normalization here we have $\alpha_{p,0}^2\alpha_{p,1}\alpha_{p,2} = 1$. For $p \mid N$, the pth Euler factors defining the Andrianov Spinor L-function are given by

$$L_p(s, F, \mathrm{Spin}) = (1 - \lambda_F(p)p^{-s})^{-1}$$

where $\lambda_F(p)$ now refers to the eigenvalue of the Frobenius operator acting on F as defined in [A01]. Again, our normalization of the L-function here means our $\lambda_F(p)$ differs from Andrianov's by a factor of $p^{k-3/2}$. The Spinor L-function satisfies the functional equation given by

$$\Lambda_F(s) = (-1)^k \Lambda_F(1-s)$$

where

$$\Lambda_F(s) = (2\pi)^{-2(s+k-3/2)}\Gamma(s+k-3/2)\Gamma(s+1/2)L(s,\pi_F,\mathrm{Spin}).$$

One has that $L(s,\pi_F,\mathrm{Spin})$ has meromorphic continuation to $\mathbb{C}$ with at most simple poles at $s=3/2$ and $s=-1/2$.

Given any Euler product of the form $L(s,X)=\prod_p L_p(s,X)$, we write $L^{(N)}(s,X) := \prod_{p\nmid N} L_p(s,X)$ and $L_{(N)}(s,X) := \prod_{p\mid N} L_p(s,X)$.

3. CAP forms

In this section we give the relevant definitions and results generalizing those of $F \in \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ to the case of level $\Gamma_0^2(N)$ for $N>1$. For a more detailed exposition of the material in this section the reader is urged to consult [PS08] or [PS].

Let $\mathrm{P}=\mathrm{M}\,\mathrm{N}$ be a proper parabolic subgroup of $\mathrm{G}(\mathbb{A})$ where M is the Levi subgroup. Let τ be an irreducible cuspidal automorphic representation of M. A cuspidal automorphic represetation π of $\mathrm{G}(\mathbb{A})$ is said to be CAP (cuspidal associated to parabolic) if there is an irreducible component π' of $\mathrm{Ind}_{\mathrm{P}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}\tau$ so that $\pi_p \cong \pi'_p$ for almost all places p. Our interest in CAP forms is that they provide the natural generalization of Saito-Kurokawa lifts when we consider $N>1$. In particular, if π_F is CAP then it must be CAP to the Siegel parabolic ([PS], Corollary 4.5). If $N=1$ then π_F is CAP if and only if F is a classical Saito-Kurokawa lifting. Suppose now that $N>1$. More generallly one has π is CAP if and only if it is a theta lift or a theta lift twisted by an idele class character. We know from [P83] that if $F \in \mathcal{S}_k^{\mathrm{M}}(\Gamma_0^2(N))$ then it is a theta lift and so CAP forms are a generalization of Saito-Kurokawa lifts. In general one has that π is a theta lift if and only if $L(s,\pi,\mathrm{Spin})$ has a pole ([P83].) If π is a twist of a theta lift by a non-trivial character then $L(s,\pi,\mathrm{Spin})$ has no poles. One should observe that since we are assuming F is without character, we can say that π_F is CAP if and only if it is either a theta lift or a twist of a theta lift by a quadratic character. In such a case, we have the following characterization of the local representations $\pi_{F,p}$ for $p\nmid N$ (see [PS08].)

1. If π_F is a theta lift, then for $p\nmid N$ the local representation $\pi_{F,p}$ is the spherical constituent of the induced representation of the form $\nu^{1/2}\chi \times \nu^{1/2}\chi^{-1} \rtimes \nu^{-1/2}$ with $|\chi|=1$ and ν the normalized absolute value from $\mathbb{Q}_p^\times \to \mathbb{C}^\times$.
2. If π_F is the twist of a theta lift by a quadratic character $\sigma_0 = \otimes\sigma_{0,p}$, then for each $p\nmid N$ for which $\sigma_{0,p}$ is unramified, the local representation is the spherical constituent of the induced representation of the form $\nu^{1/2}\chi \times \nu^{1/2}\chi^{-1} \rtimes \nu^{-1/2}\sigma_{0,p}$ with $|\chi|=1$.

The following theorem is essentially Theorem 3.1 of [PS08]. The second part of the theorem is not stated there, but it is easily deduced from their arguments. We include a proof for the reader's convenience.

Theorem 3.1. *Let N and k be positive integers with $k > 2$. Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a non-zero Hecke eigenform with eigenvalues $\lambda_F(n)$ for all n with $\gcd(n, N) = 1$. Let $\pi_F = \otimes \pi_{F,p}$ be the corresponding irreducible cuspidal automorphic representation of $\mathrm{G}(\mathbb{A})$.*

(1) *If π_F is a theta lift, then for all $p \nmid N$ and all $n > 0$ we have $\lambda_F(p^n) > 0$.*
(2) *Suppose π_F is a twist of a theta lift by a non-trivial quadratic character σ_0. For those primes where $\sigma_0(p) = 1$ we have that $\lambda_F(p^n) > 0$ for all $n > 0$. For those primes where $\sigma_0(p) = -1$, we have that $\lambda_F(p^n) > 0$ for n even and $\lambda_F(p^n) < 0$ for n odd.*

Proof. Proposition 4.1 of [PS] shows that if $\pi_{F,p}$ is given by $\chi_1 \times \chi_2 \rtimes \sigma$, then for $n > 0$ one has

$$\frac{\lambda_F(p^n)}{(p^n)^{k-3/2}} = A_{a,b}(n) + (1 - 1/p) \sum_{j=1}^{[n/2]} A_{a,b}(n - 2j) \tag{1}$$

where

$$A_{a,b}(m) = \left(\sum_{j=0}^{m} a^{m-j} b^j \right) \left(\sum_{j=0}^{m} (ab)^{-j} \right)$$

where $a = \sigma(p)$ and $b = \sigma(p)\chi_1(p)$.

In the situation of π_F being a theta lift, using the characterization of $\pi_{F,p}$ given above we have that $a = \nu(p)^{-1/2} = p^{1/2}$ and $b = \chi(p)$. Thus,

$$A_{a,b}(m) = p^{m/2} \left| \sum_{j=0}^{m} (p^{1/2}\chi(p))^{-j} \right|^2 > 0.$$

Hence, we see that $\lambda_F(p^n) > 0$ for all $n > 0$ and $p \nmid N$.

Suppose now that π_F is the twist of a theta lift by a non-trivial quadratic character σ_0. In this case we obtain that $a = p^{1/2}\sigma_{0,p}(p)$ and $b = \chi(p)\sigma_{0,p}(p)$. From this we calculate that

$$A_{a,b}(m) = (p^{1/2}\sigma_{0,p}(p))^m \left| \sum_{j=0}^{m} (p^{1/2}\chi(p))^{-j} \right|^2 .$$

From this it is clear that if $\sigma_{0,p}(p) = 1$ then $A_{a,b}(m)$ is positive for all m used in equation (1) and so $\lambda_F(p^n) > 0$. If $\sigma_{0,p}(p) = -1$, then we see that $A_{a,b}(m) > 0$ for m even and $A_{a,b}(m) < 0$ for m odd. Equation (1) then clearly gives the result. □

Remark 3.2. One should note that it is necessary to remove the eigenvalues $\lambda_F(p)$ for $p \mid N$ in the above theorem. For example, if $F \in \mathcal{S}_k^{\mathrm{M}}(\Gamma_0^2(N))$ is a Saito-Kurokawa lift of $f \in S_{2k-2}(\Gamma_0(N))$, then $\lambda_F(p) = \lambda_f(p)$ for all $p \mid N$. In this case it is entirely possible that $\lambda_F(p) < 0$ for $p \mid N$. For example, if $N = 11$ and $k = 3$, then the dimension of $S_4(\Gamma_0(11))$ is 2 and each newform has $\lambda_f(11) < 0$. For $N = 7$ and $k = 7$ we have that the dimension of $S_7(\Gamma_0(7))$ is dimension 7 and one has newforms with $\lambda_f(7) > 0$ and newforms with $\lambda_f(7) < 0$.

Finally, we note that in order to apply the results of Pitale-Schmidt it is necessary to assume $k > 2$. This follows from the fact that their argument relies in an essential way on a result of Chai-Faltings that requires $k > 2$. See Proposition 3.3 of [PS] for the precise statement they need. However, one can follow the same argument as given in [B99] (ignoring those primes $p \mid N$) to conclude the following proposition which as least gives the "easy" direction of the result for $k \geq 2$. The "hard" direction would require a classification of the local representations $\pi_{F,p}$ that can occur in the case $k = 2$.

Proposition 3.3. *Let $F \in \mathcal{S}_k^{\mathrm{M}}(\Gamma_0^2(N))$ be a non-zero Hecke eigenform with eigenvalues $\lambda_F(n)$. Then we have $\lambda_F(n) > 0$ for all n with $\gcd(n, N) = 1$.*

4. Non-CAP forms

Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be such that π_F is not CAP. Let F have Satake parameters $\alpha_{p,0}, \alpha_{p,1}$ and $\alpha_{p,2}$ as in § 2. The Ramanujan-Petersson conjecture states that

$$|\alpha_{p,1}| = |\alpha_{p,2}| = 1$$

for all $p \nmid N$. A proof of this conjecture has been announced in [W93] and we assume its validity throughout this section. We assume F is a newform, where we take the definition of newform given in [A00]. We closely follow the arguments of [KS07] in this section. The goal of this section is to prove the following theorem.

Theorem 4.1. *Let $N > 1$ and let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a non-zero Siegel newform such that π_F is not CAP. There exists a positive integer n with*

$$n \ll k^2 \log^{20} k$$

such that $\lambda_F(n) < 0$.

Write

$$L(s, \pi_F, \mathrm{Spin}) = \sum_{n \geq 1} a_F(n) n^{-s}$$

and

$$L(s, \pi_F, \mathrm{Spin}) = L_{(N)}(s, \pi_F, \mathrm{Spin}) \sum_{n \geq 1} b_F(n) n^{-s}$$

where we recall $L_{(N)}(s, \pi_F, \mathrm{Spin}) = \prod_{p \mid N} L_p(s, \pi_{F,p}, \mathrm{Spin})$. From our normalization of the Satake parameters along with the Ramanujan-Petersson conjecture we see that

$$|b_F(n)| \leq d_4(n)$$

where

$$\zeta^4(s) = \sum_{n \geq 1} d_4(n) n^{-s}.$$

We have that $L_{(N)}(1, \pi_F, \mathrm{Spin})$ is bounded as it is a finite product and $L(s, \pi_F, \mathrm{Spin})$ has analytic continuation as π_F is not CAP. We combine this with the fact that $\zeta^4(s)$ has a pole of order 4 at $s = 1$ to conclude by a standard Tauberian argument that

$$\sum_{x_0 \leq n \leq x} |a_F(n)| \ll_{x_0} x \log^3 x$$

for $x_0 > 1$. From this estimate we conclude exactly as in [KS07] that we have:

Proposition 4.2. *For $c > 1$ we have*

$$|L(c + it, \pi_F, \mathrm{Spin})| \ll 1 + \frac{c}{(c-1)^4}$$

for all $t \in \mathbb{R}$.

One can see Proposition 1 of [KS06] for a proof of this type of result in a slightly different setting. We can apply the same argument as in (Page 56-57, [KS07]) to conclude the following result.

Proposition 4.3. *Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a non-zero newform such that π_F is not CAP. Let $0 < \delta < 1/2$. Then for all $t \in \mathbb{R}$ we have*

$$|L(\delta + it, \pi_F, \mathrm{Spin})| \ll k^{1-\delta} \log^4 k \left| 1 + \frac{1}{2 \log k} + \delta + it \right|^{2 - 2\delta + 1/\log k}. \tag{2}$$

We obtain following generalization of Proposition 2 of [KS07].

Proposition 4.4. *Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a non-zero newform such that π_F is not CAP. We have*

$$\sum_{\substack{n \leq x \\ \gcd(n,N)=1}} \lambda_F(n) \log^2\left(\frac{x}{n}\right) \ll (k \log^8 k) x^{\frac{2}{3 \log k}}.$$

Proof. This is essentially the same proof as in [KS07] combined with Proposition 4.3. The only difference in this case is in the application of Perron's formula one must use $\zeta^{(N)}(2s+1)^{-1} L^{(N)}(s, \pi_F, \mathrm{Spin})$ instead of $\zeta(2s+1)^{-1} L(s, \pi_F, \mathrm{Spin})$. One then uses that $\zeta_{(N)}(2s+1)$ and $L_{(N)}(s, \pi_F, \mathrm{Spin})^{-1}$ are both absolutely bounded for $s = \delta + it$ with $\delta = \frac{2}{3 \log k}$ to achieve the same bound as in Proposition 2 of [KS07]. □

Finally, we give a lower bound for the sum of eigenvalues.

Proposition 4.5. *Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a non-zero newform such that π_F is not CAP. Suppose that $\lambda_F(n) \geq 0$ for $1 \leq n \leq x$ with $\gcd(n, N) = 1$. Then we have*

$$\sum_{\substack{n \leq x \\ \gcd(n,N)=1}} \lambda_F(n) \log^2\left(\frac{x}{n}\right) \gg \frac{\sqrt{x}}{\log^2 x}.$$

Proof. It is straightforward to show that

$$\sum_{\substack{n\leq x\\ \gcd(n,N)=1}} \lambda_F(n)\log^2\left(\frac{x}{n}\right) \gg \sum_{\substack{n\leq x/2\\ \gcd(n,N)=1}} \lambda_F(n).$$

Thus, if we can show that

$$\sum_{\substack{n\leq x\\ \gcd(n,N)=1}} \lambda_F(n) \gg \frac{\sqrt{x}}{\log^2 x}$$

we will be done.

For $p\nmid N$ we have

$$L_p(X,F,\mathrm{Spin})^{-1} = 1-\lambda_F(p)X+(\lambda_F(p)^2-\lambda_F(p^2)-1/p)X^2-\lambda_F(p)X^3+X^4$$

and for $p\mid N$ we have

$$L_p(X,F,\mathrm{Spin})^{-1} = 1-\lambda_F(p)X.$$

We can use the fact that

$$\zeta(2s+1)^{-1}L(s,F,\mathrm{Spin}) = \sum_{n\geq 1}\lambda_F(n)n^{-s}$$

to conclude that for all p we have

$$(1-X^2/p)L_p(X,F,\mathrm{Spin}) = \sum_{n\geq 0}\lambda_F(p^n)X^n.$$

Thus, for $p\nmid N$ we have

$$\begin{aligned}\lambda_F(p^n) =&\lambda_F(p)\lambda_F(p^{n-1})-(\lambda_F(p)^2-\lambda_F(p^2)-1/p)\lambda_F(p^{n-2})\\ &+\lambda_F(p)\lambda_F(p^{n-3})-\lambda_F(p^{n-4})\end{aligned} \tag{3}$$

where we put $\lambda_F(p^n)=0$ for $n<0$ and for $p\mid N$ we have

$$\begin{aligned}\lambda_F(p^2) &= \lambda_F(p)^2-1/p,\\ \lambda_F(p^n) &= \lambda_F(p)\lambda_F(p^{n-1}) \quad (n>2).\end{aligned}$$

With our normalization of the Satake parameters, the Ramanujan-Petersson conjecture, and equation (3) we have

$$|\lambda_F(p)|, |\lambda_F(p^2)|, |\lambda_F(p^3)| \ll 1. \tag{4}$$

Let $S=\{p : p\nmid N, p\leq \sqrt[4]{x}\}$. Since we are assuming that $\lambda_F(n)\geq 0$ for $n\leq x$ with $\gcd(n,N)=1$, we have

$$\sum_{\substack{n\leq x\\ \gcd(n,N)=1}} \lambda_F(n) \geq \sum_{p,q\in S}\lambda_F(p^2q^2)+\sum_{p,q\in S}\lambda_F(p^2q)+\sum_{p,q\in S}\lambda_F(pq). \tag{5}$$

As is shown in [KS07], for $p \in S$ we have

$$\lambda_F(p^4) \gg \lambda_F(p^2)^2 - c_1$$
$$\lambda_F(p^3) \gg \lambda_F(p)\lambda_F(p^2) - c_2$$
$$\lambda_F(p^2) \gg \lambda_F(p)^2 - c_3$$

where c_1, c_2 and c_3 are absolute constants each greater then 0. Let $\pi(x)$ denote the number of primes $p \leq x$ for any $x > 1$. Then we have

$$\sum_{p,q\in S} \lambda_F(p^2q^2) \gg \left(\sum_{p\in S} \lambda_F(p^2)\right)^2 - c_1\pi(\sqrt[4]{x}),$$
$$\sum_{p,q\in S} \lambda_F(p^2q) \gg \left(\sum_{p\in S} \lambda_F(p^2)\right)\left(\sum_{p\in S} \lambda_F(p)\right) - c_2\pi(\sqrt[4]{x}),$$
$$\sum_{p,q\in S} \lambda_F(pq) \gg \left(\sum_{p\in S} \lambda_F(p)\right)^2 - c_3\pi(\sqrt[4]{x}).$$

Combining these equations with equation (5) we obtain

$$\sum_{\substack{n\leq x\\ \gcd(n,N)=1}} \lambda_F(n) \gg \left(\sum_{p\in S} \lambda_F(p^2) + \sum_{p\in S} \lambda_F(p)\right)^2 - c\pi(\sqrt[4]{x}) \tag{6}$$

with $c > 0$ an absolute constant.

We claim that there exists an absolute constant $d > 0$ so that for any $p \in S$ we have

$$\lambda_F(p^2) + \lambda_F(p) \geq d.$$

Suppose not. By assumption $\lambda_F(p^2)$ and $\lambda_F(p)$ are both greater than or equal to 0, so we must have that $\lambda_F(p^2)$ and $\lambda_F(p)$ are both small. Equation (3) gives

$$\lambda_F(p^3) = \lambda_F(p)\lambda_F(p^2) - (\lambda_F(p)^2 - \lambda_F(p^2) - 1/p)\lambda_F(p) + \lambda_F(p),$$

and so $\lambda_F(p^3)$ must be small as well. However, equation (3) also shows that $\lambda_F(p^4)$ is given by

$$\lambda_F(p^4) = (\lambda_F(p^2)^2 + \lambda_F(p)\lambda_F(p^3) + \lambda_F(p^2)(1/p - \lambda_F(p)^2) + \lambda_F(p)^2) - 1.$$

This contradicts $\lambda_F(p^4) \geq 0$ if $\lambda_F(p^2)$ and $\lambda_F(p)$ are arbitrarily small. Thus, such a $d > 0$ exists. We combine this fact with equation (6) along with the prime number theorem to conclude that

$$\sum_{\substack{n\leq x\\ \gcd(n,N)=1}} \lambda_F(n) \gg \frac{\sqrt{x}}{\log^2 x}.$$

□

Combining the previous two propositions we see that if $F \in \mathcal{S}_k(\Gamma_0^2(N))$ is a newform such that π_F is not CAP and $\lambda_F(n) \geq 0$ for $n \leq x$ with $\gcd(n, N) = 1$ we have

$$\frac{\sqrt{x}}{\log^2 x} \ll (k \log^8 k) x^{\frac{2}{3 \log k}}. \tag{7}$$

However, this equation cannot hold for large enough x. In particular, following [KS07] we see that for equation (7) to hold we must have

$$x \ll k^2 \log^{20} k,$$

which finishes the proof of Theorem 4.1.

Remark 4.6. For $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ a Hecke eigenform, it was shown in [K07] that $\lambda_F(n)$ changes sign infinitely often. This result has been generalized in [PS08] to the case $N > 1$. Their result shows that if F is a Hecke eigenform for all $T(n)$ with $\gcd(n, N) = 1$ such that π_F is not CAP, then there exists an infinite set S_F of primes numbers $p \nmid N$ such that if $p \in S_F$, then there are infinitely many r such that $\lambda_F(p^r) > 0$ and infinitely many r such that $\lambda_F(p^r) < 0$.

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