Sums of Three Squares:

Cogdell

$\mathbb{Q} \supseteq \mathbb{Z} \ni m$

$m = l_1^2 + l_2^2$ Fermat, Euler

$m = l_1^2 + l_2^2 + l_3^2$ Legendre, Gauss $m \neq 4^a(7+8b)$ then one can.

$m = l_1^2 + l_2^2 + l_3^2 + l_4^2$ Lagrange

$l_i \in \mathbb{Z}$

$q(\vec{x}) = \sum a_{ij} x_i x_j \quad a_{ij} \in \mathbb{Z}$ Can read about this in Serre's book.

$m = q(\vec{l})$?

Number Fields: $K \supseteq \mathbb{Q}$

$K \supset \mathcal{O} \supset \wp$

$V = K^n \supset L = \mathcal{O}^n$

$q(\vec{x}) = \sum a_{ij} x_i x_j \quad a_{ij} \in \mathcal{O}$

$\alpha \in \mathcal{O}$ $\boxed{\text{When can one write } \alpha = q(\vec{l}),\ l \in \mathcal{O}^n ?}$

I.R. : Integral representability ?

$n = 2$: Hilbert Class Field Theory

$n \geq 3$: Hilbert's 11th problem

stated around 1900

History and Results:

1. Siegel (1930's)

Eg. $\mathbb{Q} \supseteq \mathbb{Z}$ $\quad m = l_1^2 + l_2^2 + l_3^2$

$$\Rightarrow \text{Can solve } \begin{cases} m \equiv l_1^2 + l_2^2 + l_3^2 \pmod{p^f} & \text{for all } p, f \\ m > 0 \end{cases}$$

necessary conditions for I.R.

Are these sufficient?

Q1: Given $\alpha \in \mathcal{O}$ can we solve the congruences

$\alpha \equiv q(\vec{l}) \bmod \wp^f$ all $\wp, f$ (including $\wp | \infty$)

or Is $\alpha = q(\vec{l})$ solvable $\vec{l} \in L_\wp$ $\wp$-adic completion of L.

(L.I.R = local integral representability)

Q2: When is I.R $\Longleftrightarrow$ L.I.R ?

1930's: Siegel solved L.I.R definitively and quantitatively.

Siegel 1950's If $q(\vec{x})$ is indefinite and $n \geq 4$ then I.R $\Longleftrightarrow$ L.I.R.

(indefinite means no conditions on $\wp | \infty$)

Kneser, Hsia 1960's - 1970's $\quad q(x)$ indef., $n = 3$ $\quad$ L.I.R $\Longleftrightarrow$ I.R

(completely different techniques, algebraic techniques)

So we are left to K totally real ($K \hookrightarrow \mathbb{R} \subseteq \mathbb{C}$)

and $q(x)$ totally positive definite.

1978: Hsia, Kitaoka, Kneser : If $n \geq 5$, $\exists\, c > 0$ effective, s.t.

if $\alpha \in \mathcal{O}$ is totally positive, $N(\alpha) > c$, then I.R $\Leftrightarrow$ L.I.R.

If $n = 4$: Essentially the same w/ an extra condition or two on α.

$n = 3$: What remained.

Theorem (C., Piatetski-Shapiro, Sarnak): If $n = 3$, $q(\vec{x})$ tot. pos. integral quad. form, then $\exists\, c > 0$, ineffective, s.t. α tot. positive, $N(\alpha) > c$, α square free then I.R $\Leftrightarrow$ L.I.R.

Restrict to $K = \mathbb{Q}$, $V = \mathbb{Q}^3 \supseteq \mathbb{Z}^3$, $q(\vec{x}) = x_1^2 + x_2^2 + x_3^2$.

Generating functions I : Modular forms

$$\mathfrak{H} = \{ z = x + iy : y > 0 \}$$

$$\vartheta(z, L) = \sum_{\ell \in L} e^{2\pi i q(\vec{\ell}) z} = 1 + \sum_{m=1}^{\infty} r(m, L) e^{2\pi i m z}$$

$$r(m, L) = \#\{ \vec{\ell} \in L : q(\vec{\ell}) = m \}$$

I.R. $\Leftrightarrow$ when is $r(m, L) \neq 0$.

Local question?

Two lattices L, L' are in the same genus if L.I.R problem are the same, or $L_\wp \simeq L'_\wp$.

Siegel attached a ϑ-series to the genus

$$\vartheta(z, \mathrm{Gen}(L)) = \frac{\displaystyle\sum_{[L'] \in \mathrm{Gen}(L)} \frac{\vartheta(z,L')}{o(L')}}{\displaystyle\sum_{[L'] \in \mathrm{Gen}(L)} \frac{1}{o(L')}}$$

$$o(L') = \# \left\{ \gamma \in O(q) : \gamma L = L' \right\}.$$

$$\vartheta(z, \mathrm{Gen}(L)) = 1 + \sum_{m=1}^{\infty} r(m, \mathrm{Gen}(L)) e^{2\pi i m z}$$

L.I.R $\iff$ $r(m, \mathrm{Gen}(L)) \neq 0$.

$\vartheta(z,L)$ and $\vartheta(z, \mathrm{Gen}(L))$ are modular forms.

$\exists$ $\Gamma \subseteq SL_2(\mathbb{Z})$ s.t.

$$\vartheta(\gamma z, L) = \mu(\gamma)(cz+d)^{3/2} \vartheta(z,L) \quad \in M_{3/2}(\Gamma)$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \ \gamma z = \frac{az+b}{cz+d} \qquad \text{all } \gamma \in \Gamma$$

Since $\vartheta(z, \mathrm{Gen}(L))$ is just a weighted sum of modular forms, it is a modular form as well. The ϑ-series is modular essentially due to Poisson summation. It is not mysterious that these are modular forms.

Siegel: $\vartheta(z, \mathrm{Gen}(L)) = E(z)$ Eisenstein series

$$r(m, \mathrm{Gen}(L)) = \prod_p \delta_p(m, L_p) \underset{\text{if} \neq 0}{\gg} m^{1/2-\varepsilon}$$

compute these explicitly in δ_p : local mass densities

very robust.

$$\left(\delta_2(m, L_2) \neq 0 \iff m \neq 4^a(7+8b) \right)$$

This gives the solution to L.I.R.

For the other problem, we need to know about

$$\vartheta(z, L) - \vartheta(z, \mathrm{Gen}(L)) = \tilde{f}(z) \in S_{3/2}(\Gamma).$$

cusp form means it vanishes as you go out to a cusp

$$\tilde{f}(z) = \sum_{m=1}^{\infty} a_{\tilde{f}}(m)\, e^{2\pi i m z}$$

$$\underset{\text{want}}{r(m, L)} = \underbrace{r(m, \mathrm{Gen}(L))}_{m^{1/2-\varepsilon}} + \underbrace{a_{\tilde{f}}(m)}_{\text{"error"}}$$

$$S_{3/2}(\Gamma) = S'_{3/2}(\Gamma) + \boxed{S^{\circ}_{3/2}(\Gamma)}$$

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understand

assoc to quad. forms of 1 variable

coeffs. are robust X about same size as E.S., but they are sparse.

if m is large enough and sq. free we avoid these.

For $\tilde{f} \in S_{3/2}(\Gamma)$ we expect (Ramanujan conj)

$$|a_{\tilde{f}}(m)| \leq m^{1/4+\varepsilon}.$$

How do we control these coeff.?

<u>Generating Functions II : L-functions</u>

These $\tilde{f} \in S^{\circ}_{3/2}(\Gamma) \xleftrightarrow[\Theta\text{-Lift}]{\text{Shimura}} \varphi(z) \in S_2(\Gamma')$ companion form

$$\varphi(z) = \sum_{n=1}^{\infty} a_\varphi(n) e^{2\pi i n z}$$

$$L(s,\varphi) = \sum_{n=1}^{\infty} \frac{a_\varphi(n)}{n^{s+1/2}}$$

- Converges for $\operatorname{Re}(s) > 1$
- entire continuation to $\mathbb{C}$
- $\Gamma(s+1/2) L(s,\varphi) \varepsilon(s) = \Lambda(s,\varphi) \sim \Lambda(1-s,\varphi)$

F, E $s \leftrightarrow 1-s$.

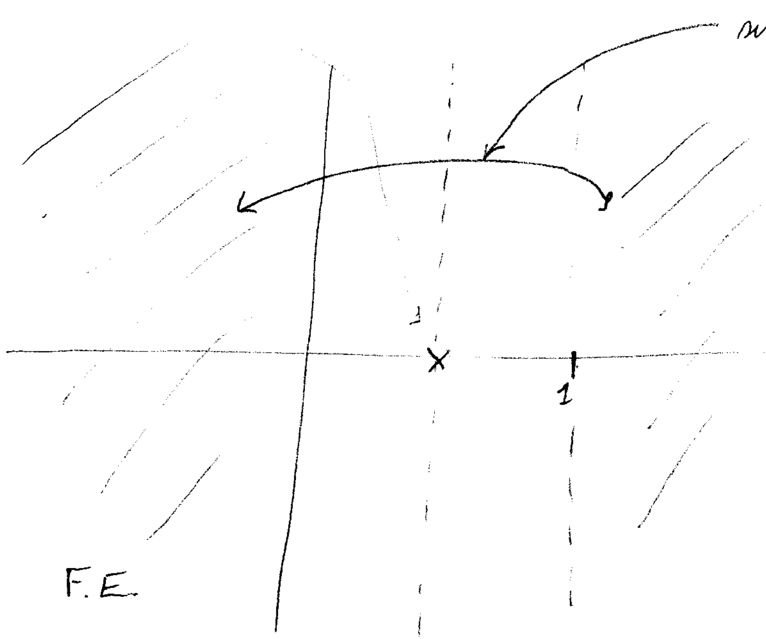
Theorem (Waldspurger, Shimura, Baruch-Mao):

$$|a_{\tilde{\varphi}}(m)|^2 \approx m^{1/2} L(1/2, \varphi \times \chi_m)$$

$$\chi_m(p) = \begin{cases} 1 & p = \mathfrak{p}_1\mathfrak{p}_2 \\ -1 & (p) = \mathfrak{p} \\ 0 & p \mid m \end{cases} \qquad \mathbb{Q}(\sqrt{m})/\mathbb{Q}$$

$$L(s, \varphi \times \chi_m) = \sum \frac{a_\varphi(n)\chi_m(n)}{n^{s+1/2}}$$

since we can control both sides, Phragmen-Lindelöf convexity



ab. conv.

Need: $L(1/2, \varphi \times \chi_m) \ll m^{1/2-\delta}$ any fixed $\delta > 0$.

Convexity: $L(1/2, \varphi \times \chi_m) \ll m^{1/2+\varepsilon}$

Need: any subconvex estimate

Thm: $L(\frac{1}{2}, \varphi \times \chi_m) \ll m^{\frac{1}{2} - \frac{7}{130} + \varepsilon}$.

Cor: $|a_{\phi}(m)| \ll m^{\frac{1}{2} - \frac{7}{260} + \varepsilon}$

Cor: I.R $\Longleftrightarrow$ L.I.R under condition of theorems stated earlier.