Motivic cohomology actions and the geometry of eigenvarieties:

Joint work w/ Jack Thorne.

Let G be a connected reductive group / $\mathbb{Q}$.

$G_\infty = G(\mathbb{R}) \supseteq K_\infty$ = max compact-mod center subgroup.

$D_\infty = G_\infty / K_\infty$ usual symmetric space.

$\ell_0 = \mathrm{rank}(G_\infty) - \mathrm{rank}(K_\infty)$

$q_0 = \frac{1}{2}(\dim D_\infty - \ell_0)$

For any $K \subset G(\mathbb{A}_f)$ open compact;

$$Y_K = G(\mathbb{Q}) \backslash (D_\infty \times G(\mathbb{A}_f))/K$$

$$= \coprod_{i \text{ fin.}} (\Gamma_{i,K} \backslash D_\infty) .$$

Observation: The integer ℓ_0 (& to a lesser extent, q_0 as well) controls many aspects of the geometry and arithmetic of G and of the Y_K's.

1. (Harish-Chandra) G^{ad}_∞ has a discrete series iff $\ell_0 = 0$.
   If D_∞ is Hermitian symmetric, $\ell_0 = 0$.

2. (Bergeron-Venkatesh) The cohomology $H^*(Y_K, \mathbb{Z})$ has a lot of torsion iff $\ell_0 = 1$. (This is a conjecture.)

3. (Borel-Wallach, Zuckerman) Let $\mathcal{L}_{\lambda,\mathbb{C}}$ be an irreducible algebraic representation of $G(\mathbb{C})$. Then the tempered cusp forms on G contribute to $H^n(Y_K, \mathcal{L}_{\lambda,\mathbb{C}})$ exactly for $n \in [q_0, q_0 + \ell_0]$, and

the π-part of $H^n(Y_K, \mathcal{L}_{\lambda,\mathbb{C}})$ has dimension $m_{\pi,K} \cdot \binom{l_0}{n-q_0}$

for $q_0 \le n \le q_0 + l_0$. This is a purely archimedean calculation.

4. Let π be a tempered regular algebraic cusp form on G: Then one conjectures $L(s, \mathrm{ad}\,\pi)$ has order of vanishing at $s=0$ equal to l_0.

 This is known for $GL(n)$, but one must use deep work of Jacquet, Shalika, Piatetski-Shapiro, etc.

5. (Hida, Urban) Let $\mathfrak{X} = \mathfrak{X}_{G,K}$ be the eigenvariety for G (of some tame level K), with its weight map $\omega : \mathfrak{X} \to \mathcal{W} = \mathcal{W}_{G,K}$. Then any component of $\mathfrak{X}$ containing a point associated with a (regular algebraic tempered) cusp form on G has dimension conjecturally $\dim(\mathcal{W}) - l_0$.

 "Nonabelian Leopoldt conjecture"

<u>Goal</u>: • Discuss an "arithmetic enhancement" of 3. as conjectured by Venkatesh (& motivated using 4.)

• (our contribution) Discuss a surprising link between Venkatesh's conjecture and 5.

From now on π is always regular algebraic tempered.

Take $G = \mathrm{Res}_{F/\mathbb{Q}} GL_n$ for F a number field with r_1, r_2 the number real and complex places respectively for the rest of the talk.

In this case, $l_0 = \lfloor \frac{n-1}{2} \rfloor r_1 + (n-1) r_2$

$\rightsquigarrow$ $l_0 = 0$ iff $n = 2$ & F tot. real.

Let π be as before. A conjecture of Clozel says there should be a "motive" M_π canonically assoc. with π, defined over the field of coeffs $\overset{E}{\overset{\shortparallel}{E_\pi}}$ given by Hecke e.v.s of π.

If F is totally real or CM, the compatible system of Galois reps. which M_π would give rise to exist unconditionally (Eichler, Shimura, ..., HLTT, Scholze, Boxer). ~~conditionally~~

Note: $H^*(Y_K, \mathcal{L}_{\lambda, E})_\pi$ is perfectly well-defined; is there a "purely arithmetic" explanation of the dimensions of the $H^n(Y_K, \mathcal{L}_{\lambda,E})_\pi$'s?

Conj: (Venkatesh): There is a canonical E-vector space L_π of dim. l_0 together with a canonical action of $\Lambda_E^* L_\pi$ on $H^*(Y_K, \mathcal{L}_{\lambda,E})_\pi$ which makes the latter finite free and gen. in degree $l_0 + q_0$ over this Λ-alg, and $L_\pi = \mathrm{Ext}^1_{\mathcal{M}Mot/\mathbb{Q}_F}(\mathbb{1}, (\mathrm{ad}\, M_\pi)(1))$.

Why? If you believe in a motivic paradise, then $\dim_E L_\pi = l_0$.

Goal: Understand this conjecture "$\otimes \mathbb{Q}_p$".

Why should $\otimes \mathbb{Q}_p$ make things more accessible?

- M_π becomes $\rho_\pi : \mathrm{Gal}(\overline{F}/F) \to GL_n(L)$ (here $L/\mathbb{Q}_p$ is a suitable finite extension)
- L_π becomes $H^1_f(F, \mathrm{ad}(\rho_\pi)(1))$ = Bloch-Kato Selmer group.

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Fix $F = \mathbb{Q}$ ($l_0 = \lfloor \frac{n-1}{2} \rfloor$) and let π be as before. Fix p, L, $\rho_\pi : Gal(\bar{F}/F) \to GL_n(L)$. Assume:

- π_p is unramified and has regular s.s. Satake parameters
- π_∞ satisfies a parity condition if n is odd.

Choose an ordering α on the eigenvalues of the Satake parameters of π_p. ($n=2$ this is choosing a p-stabilization) which satisfies a "small slope" condition.

$\rightsquigarrow$ The pair (π, α) defines a point $x = x(\pi, \alpha) \in \mathcal{X} = \mathcal{X}_{GL_n/\mathbb{Q}, K, (N_\pi)}$
$\lambda = w(x) \in \mathcal{W}$.

__Theorem:__ Let $\mathbb{T}_x = \hat{\mathcal{O}}_{\mathcal{X},x}$ and $\Lambda = \hat{\mathcal{O}}_{\mathcal{W},\lambda}$ so these are local Noetherian L-algebras and $\mathbb{T}_x$ is a finite Λ-algebra.
Then $\dim \mathbb{T}_x \geq \dim \mathcal{W} - l_0$, and if equality holds then.

1. The map $\Lambda \to \mathbb{T}$ is surjective and $\mathbb{T}_x$ is a complete intersection.
2. Let $V_x = (\ker \Lambda \to \mathbb{T}_x) \otimes_\Lambda L$, then there is a canonical action of $\Lambda_L^* V_x$ on $H^*(Y_{K,(N_\pi p)}, \mathcal{L}_{\lambda,L})_{(\pi,\alpha)}$.
$$H^*(Y_{K,(N_\pi p)}, \mathcal{L}_{\lambda,L})_{(\pi,\alpha)} =: H^*_{\pi,\alpha}$$
s.t. $H^*_{\pi,\alpha}$ is free of rank one over $\Lambda_L^* V_x$.
3. If one believes a suitable "$R = \mathbb{T}_x$ conjecture", then $V_x \cong H^1_f(\mathbb{Q}, ad(\rho_\pi)(1))$.
canonically.

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Idea of proof: 1. Define a canonical $\mathbb{T}_x$-module H_x so that $\mathrm{Tor}_i^{\Lambda}(H_x, L) \cong H_{\pi,\alpha}^{q_0+l_0-i}$

(follows from the construction of $\mathcal{X}$ & a spectral seq.)

By construction, H_x is a faithful $\mathbb{T}_x$ module, but $H_x \otimes_\Lambda L \simeq H_{\pi,\alpha}^{q_0+l_0} \simeq L$ ~~mono~~, so $\Lambda \twoheadrightarrow H_x$.

Thus, $\mathbb{T}_x$ is a quotient of Λ. So $\Lambda \twoheadrightarrow \mathbb{T}_x \simeq H_x$.

So $\mathrm{Tor}_1^\Lambda(H_x, L) \simeq \mathrm{Tor}_1^\Lambda(\mathbb{T}_x, L)$

$\quad \cong H^{q_0+l_0-1}$

$\quad \cong (\ker \Lambda \to \mathbb{T}_x) \otimes_\Lambda L.$

so $\mathbb{T}_x$ is a complete intersection.

To get the V_x-action, use two facts

- If R comm. ring, A, B comm. R-algs, then $\bigoplus_{i \geq 0} \mathrm{Tor}_i^R(A, B)$ is can. a skew-comm. graded R-alg.
- If R is complete local Noeth, res. field L, and $I \subset R$ an ideal generated by a reg seq., then

$$\mathrm{Tor}_*^R(R/I, L) \cong \Lambda_L^*(I \otimes_R L).$$

Thus,

$$\mathrm{Tor}_*^\Lambda(\mathbb{T}_x, L) \cong \Lambda_L^* V_x \curvearrowright \mathrm{Tor}_*^\Lambda(H_x, L)$$

Remark: The "$R = \mathbb{T}_x$" conjecture involves R = trianguline deformations of ρ_π with triangulations ?... the one determined by α.

This conjecture (+NAL) implies other things.

e.g. $H^1_f(\mathbb{Q}, \mathrm{ad}\,\rho_\pi) = 0$ & $\mathcal{X}$ is smooth at $x(\pi,\alpha)$.