Automorphic Representations of $D^\times$ $\quad$ D : quaternion alg / $\mathbb{Q}$

$f : D_{\mathbb{A}}^\times \longrightarrow \mathbb{C}$

- $f(\gamma x) = f(x) \quad \forall x \in D^\times \subseteq D_{\mathbb{A}}^\times$
- $f(x z_\infty) = f(x) \quad \forall z_\infty \in Z(D^\times)_\infty$
- $f(x u_f) = f(x)$ $\quad$ $U_f \in$ compact open subgrp of $D_f^\times$
- f spans f.d. $\mathbb{C}$ v.space under right trans. $(\mathfrak{g}, K_\infty)$ (if definite it is $D_\infty^\times$)

(if D is split e.g, $D = M_2(\mathbb{Q})$, need f to be of moderate growth)

$A_0(D^\times) \cong \bigoplus_{\substack{\pi \\ \text{irred}}} \pi \leftarrow$ auto rep.

$A_0(D^\times) = \{f\}$

$D_\infty \cong \mathbb{H}$ = Hamiltonian quaternions

Now assume D definite, then $D^\times Z_\infty \backslash D_{\mathbb{A}}^\times$ is compact, so

$A_0(D^\times)$ is unitarizable wrt.

$$\langle f, g \rangle = \int_{Z_\infty D^\times \backslash D_{\mathbb{A}}^\times} f(x) \overline{g(x)}\, dx$$

One dim. reps π's are all of the form

$$D^\times Z_\infty \backslash D_{\mathbb{A}}^\times \xrightarrow{\text{norm}} \mathbb{Q}_{>0} \mathbb{R}_{>0} \backslash \mathbb{A}^\times \xrightarrow{\chi} \mathbb{C}^\times$$

$\Rightarrow$

π is cuspidal if it is orthogonal to these one dimensional reps.

§1 Yoshida Lift (due to Hiroyuki Yoshida 1980)

f_1, f_2 : on $D_{\mathbb{A}}^\times \longrightarrow$?

(in fact, this is a slight variant of def)

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$$\begin{tikzcd} f_1 \in (\pi_1 \otimes \check{U}_1)^{D_\infty^\times} & \\ \pi_{1,\infty}^\vee \arrow[u, phantom, "\rotatebox{90}{$\cong$}"] & \end{tikzcd}$$

$D^\times \times D^\times$ act on D via $(a,b)\cdot x = axb^{-1}$ and this action preserve the reduced norm up to similitudes.

$$D^\times \times D^\times \longrightarrow GO(D) = \{ g \in GL(D) : \overset{\swarrow \text{reduced norm}}{n}(g\cdot x) = \lambda(g)\, n(x) \quad \forall x \}$$

The kernel of this action is $\mathbb{Q}^\times$ embedded diagonally and has image $GO(D)^\circ$ = connected component.

$$\begin{tikzcd} \mathbb{Q}^\times \backslash D^\times \times D^\times \arrow[r, "\simeq"] \arrow[d, phantom, "\rotatebox{90}{$=$}"] & GO(D)^\circ. \\ H \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] & \\ H' = \{(a,b) : n(a) = n(b)\} \arrow[r, "\sim"] & SO(D) \end{tikzcd}$$

$$\begin{tikzcd} & (a,b) \arrow[rr, mapsto] & & f_1(a) \otimes f_2(b) & \swarrow \text{vector space} \\ f_1 \otimes f_2 : & H_{\mathbb{A}}^\times \arrow[r, "\sim"] & GO(D)^\circ_{\mathbb{A}} \arrow[r] & V & \end{tikzcd}$$

<u>Weil rep'n / reductive dual pairs</u>

$L = \mathbb{Q}w_1 \oplus \mathbb{Q}w_2$ $\qquad W = L \oplus L^\vee$ $\qquad L^\vee = \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{Q})$

has a natural alternating form

$D \otimes W =: \mathbb{W}$ $\qquad$ 16-dim. symplectic space (so has alternating form)

$\psi : \mathbb{A}/\mathbb{Q} \longrightarrow \mathbb{C}^1$

$Sp(W) \circlearrowright \{ \phi_\psi :$ Heisenberg grp. assoc to $\mathbb{W} \}$

$$\omega : \widetilde{Sp}(\mathbb{W}) \longrightarrow \overset{\text{Schrödinger model}}{\mathcal{U}(L^2(\mathbb{X}))}$$ "Weil rep'n" where

$$\mathbb{X} = \underbrace{D \otimes L^\vee}_{\mathbb{W} \cong \mathbb{X} \oplus \mathbb{X}^\vee}$$ (max isotropic subspace of $\mathbb{W}$)

$$L^2(\mathbb{X}) = L^2(\mathrm{Hom}_{\mathbb{Q}}(L, D)).$$

In our case,

$$\begin{tikzcd}
\mathbb{C}^1 \arrow[r, hook] & \widetilde{Sp}(\mathbb{W}) \arrow[r] & Sp(\mathbb{W}) . \\
& & Sp(W) \times SO(D) \arrow[u, phantom, sloped, "\subseteq"] \arrow[ul, bend left]
\end{tikzcd}$$

Thus, we obtain a rep.

$$\begin{array}{ccccc}
\omega : & Sp(W)_{/\mathbb{A}} & \times\, SO(D)_{/\mathbb{A}} & \longrightarrow & \overset{\swarrow \text{ vector-valued functions}}{\mathcal{U}(L^2(\mathbb{X})_{/\mathbb{A}})} . \\
& \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
& g & h & & \varphi_{/\mathbb{A}}
\end{array}$$

(Schwartz-Bruhat)

From φ, we obtain an automorphic form Θ_φ defined by

$$\Theta_\varphi(g,h) = \underbrace{\sum_{\mu \in \mathbb{X}_{\mathbb{Q}}} (\omega(g,h)\varphi)(\mu)}_{\substack{\rotatebox{90}{$\in$} \\ V}}$$

Θ_φ is left $Sp(W)_{\mathbb{Q}} \times SO(D)_{\mathbb{Q}}$ invariant.

So the upshot is that Θ_φ is an automorphic form on $Sp(W) \times SO(D)$.

We would like to decompose this by

$$\Theta_\varphi = \sum_{\substack{F \text{ auts on} \\ SO(D)}} Y(F) \otimes F.$$

Does this give a bijection

$$\begin{array}{ccc} F & \longrightarrow & Y(F) \\ & & \text{on} \\ \text{on} & & Sp(W) \simeq Sp_4 \\ SO(D) & & \\ \text{is} & & \\ SO(4) & & \end{array} \qquad ?$$

- If so, $g \in Sp(W)_{/\mathbb{A}}$

$$Y(F)(g) = \int_{SO(D)_{\mathbb{Q}} \backslash SO(D)_{/\mathbb{A}}} \langle \Theta_\varphi(g,h), F(h) \rangle \, dh. = \langle \Theta_\varphi(g,-), F \rangle_{SO(D)}$$

$Y(f_1 \otimes f_2)$ is called the Yoshida lift of f_1 and f_2.

- Hecke equivariant
- carries over level structure, weight, etc...

How to make this p-integral?

- compute the Fourier coefficients of $Y(f_1 \otimes f_2)$.

- Need to choose f_1, f_2 nicely
- Need to choose φ nicely

- Lift auto reps instead! π_1, π_2.

Depending on which Fourier coeff I want to compute, say the

$$T = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \qquad a, b, c \in \mathbb{Z}_{\geq 0}$$

Choose $f_i^T \in V_{\pi_i}$.

Choose $\varphi^T = \underset{v}{\otimes} \varphi_v^T$ also depending on T.

$$\Psi = \Psi(\pi_1, \pi_2) = \langle \Theta_{\varphi^T}(g, -), f_1^T \otimes f_2^T \rangle$$

Simplifies the computation and obtain an explicit formula for

$a^T(\Psi) =$ T^{th} Fourier coeff.

How to choose f^T?

$\mathbb{X} = L^\vee \otimes D$ (φ is a function on $\mathbb{X}_{\mathbb{A}}$)

$= \mathrm{Hom}_{\mathbb{Q}}(L, D)$

$\overset{\psi}{\mu}$

$$L = \mathbb{Q}\omega_1 \oplus \mathbb{Q}\omega_2 \xrightarrow{\mu} \mu_i = \mu(\omega_i) \in D$$

$$\mu \rightsquigarrow T_\mu = \begin{bmatrix} n(\mu_1) & \frac{tr(\mu_1 \bar{\mu}_2)}{2} \\ \frac{tr(\mu_1 \bar{\mu}_2)}{2} & n(\mu_2) \end{bmatrix}$$ (has to be semi-integral, ...)

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"nice" μ : image of μ is an imaginary quadratic field in D

$\mu_1 = 1$

$$\text{image}(\mu) = \mathbb{Q} \oplus \mathbb{Q}\mu_2$$

So for $T = T_\mu = \begin{bmatrix} 1 & b/2 \\ b/2 & c \end{bmatrix} \rightsquigarrow K^T$ imaginary quadratic field

$$\underset{\overset{\circlearrowright}{D}}{D} = K^T \perp j K^T \underset{\text{right } K^T \text{ v.space}}{\circlearrowleft}$$

$$D \xhookrightarrow{\varepsilon^{(T,j)}} End(K^T \oplus K^T) \cong M_2(K^T)$$

- Have rep. $\kappa \in \mathbb{Z}_{\geq 0}$ highest wt $(\kappa, -\kappa)$

$$GL_2(K^T) \overset{\sigma_\kappa}{\circlearrowright} Sym^{2\kappa} \otimes \det^{-\kappa}$$

Depending on $(T, j) \rightsquigarrow$ get a rep. of

$$D^\times \overset{\sigma_\kappa^{(T,j)}}{\circlearrowright} Sym^{2\kappa} \otimes W^{-\kappa}$$

$$e_1^\kappa \otimes e_2^\kappa$$

- $\mathfrak{D} \subseteq D$ Eichler order (level structure)

$$f_i \in \underbrace{(\pi_f)^{\mathfrak{D}_f^\times} \otimes (\pi_\infty \otimes \sigma_\kappa^{(T,j)})^{D_\infty^\times}}_{\longrightarrow \text{ 1-dimensional}}$$

"Normalize" $f_i(1) = e_1^\kappa \otimes e_2^\kappa$

(depends on choice $T, j, \mathfrak{D}$)

Similarly,

$$\varphi = \otimes \varphi_v \qquad (T, j, \mathcal{D}) \qquad \mathcal{L}^\vee =: \mathbb{Z} w_i^\vee \oplus \mathbb{Z} w_2^\vee$$

$$\varphi_v = \begin{cases} \mathrm{char}_{\mathcal{D}_\ell \otimes \mathcal{L}_\ell^\vee} & \ell < \infty \\ \left(\sum_{i=-k}^{k} P_i^k(\mu) \otimes v_i \right) e^{-2\pi \overbrace{\mathrm{tr}}^{n(\mu_1)+n(\mu_2)} (T_\mu)} & \end{cases}$$

$$v_i = e_1^{k+i} \otimes e_2^{k-i} \qquad \underset{e_1^{k-i} \otimes e_2^{k+i}}{\underset{\sim}{\mathrm{Sym}^{2k} \mathbb{C}^2}}$$

- $GL(K) \curvearrowright \mathrm{Sym}$

- $\underset{\substack{\rotatebox{90}{$\in$} \\ g}}{PGL(\mathbb{Z})} \overset{Ad}{\curvearrowright} \underset{\substack{\rotatebox{90}{$\in$} \\ X}}{M_2^{(0)}(\mathbb{Z})} = \underset{\substack{\rotatebox{90}{$\cong$} \\ \mathbb{Z}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \oplus \mathbb{Z}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \mathbb{Z}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}}{\left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right\}}$

$$Ad(g)X = gXg^{-1}.$$

$$\underset{\substack{\rotatebox{90}{$\in$} \\ g}}{Ad} \curvearrowright \underset{\substack{\rotatebox{90}{$\in$} \\ f}}{\mathcal{P}_k[a,b,c]}$$

$$(g \cdot f)(x) = f(Ad(g^{-1})\,x)$$
$$= f(g^{-1} x g)$$

Preserves a diff. operator $\Delta = \frac{\partial^2}{\partial a^2} + 4 \frac{\partial^2}{\partial b \partial c}$

$(PGL_2 \xrightarrow{\sim} SO(2,1))$

$\mathcal{H}_k[a,b,c] = \{ f \in \mathcal{P}_k : \Delta f = 0 \}$

"Harmonic polynomials"

$$\mathcal{H}_k \cong (\mathrm{Sym}^{2k})^\vee$$

$$P_i \longleftrightarrow v_i^\vee.$$

$$P_0^k = \sum_{i=0}^{\nu} (-1)^i \binom{k}{2i}\binom{2i}{i} a^{k-2i} b^i c^i \qquad \nu = \lfloor \tfrac{k}{2} \rfloor$$

On $M_2^{(0)}(\mathbb{Z})$.

! $P_0^k\left(\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}\right) = a^k$

$$P_i^k = \begin{cases} \dfrac{(k+i)!}{k!} (X^+)^{-i} P_0^k & i<0 \\ \dfrac{(k-i)!}{k!} (X^-)^{i} P_0^k & i>0 \end{cases}$$

$X^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $X^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\circlearrowleft M_2^{(0)}$ so acts on $\mathcal{H}_k$

$$P_i^k\left(\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}\right) = 0 \quad , i \neq 0.$$

$$\varphi_\infty \in \left(\left(\mathrm{Sym}^{2k} \otimes \det{}^{-k}\right) \otimes \mathcal{H}_k\right)^{PGL_2}$$

$T = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ $\qquad N = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\} \cong$ sym 2x2 matrices $\cong \mathbb{A}^3$.

$$a(T)(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\,\psi(-Tn)\,dn$$

$\psi : X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \mathbb{C}^1$

$\psi(X) = \psi_0(\mathrm{tr}(X))$.

where $\psi_0 : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^1$

$\psi_0 = \otimes_v \psi_v$

$\psi_v : \mathbb{Q}_v/\mathbb{Z}_v \to \mathbb{C}^1$

$x_v \in \mathbb{Q}_v$

$\psi_v(x_v) = e^{-2\pi i\,(\text{fractional part of } x_v)}$

$\psi_\infty(x_\infty) = e^{2\pi i (x_\infty}$

$$a(T)(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \int_{H'(\mathbb{Q})\backslash H'(\mathbb{A})} \langle \Theta_\varphi(h, ng),\ f_1 \otimes f_2(h) \rangle \, \Psi(-Tn)\, dh\, dn$$

$$a(T)(1) = \underset{\substack{\uparrow \\ \text{classical F.C.}}}{a^T} \cdot e^{\overset{?}{2\pi i\, tr(T)}}$$

Set $g = 1$.

$$\Theta_\varphi(h,n) = \sum_{\mu \in \mathbb{X}_D} (\omega(h,n)\varphi)(\mu)$$

$$= \sum_{\mu \in \mathbb{X}_D} \Psi(nT_\mu) \cdot \varphi(h^{-1}\mu)$$

$$a(T)(1) = \int_{H'(\mathbb{Q})\backslash H'(\mathbb{A})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \langle \sum_{\mu \in \mathbb{X}_D} \Psi(nT_\mu)\varphi(h^{-1}\mu),\ f_1 \otimes f_2(h) \rangle \, \Psi(-Tn)\, dn\, dh$$

$$= \int_{H'(\mathbb{Q})\backslash H'(\mathbb{A})} \langle \sum_{\mu \in \mathbb{X}_D} \varphi(h^{-1}\mu) \underbrace{\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \Psi(n(T_\mu - T))\, dn}_{\substack{=0 \text{ unless} \\ T_\mu = T.}},\ f_1 \otimes f_2(h) \rangle \, dh$$

$$= \int_{H'(\mathbb{Q})\backslash H'(\mathbb{A})} \langle \underbrace{\sum_{\substack{\mu \in \mathbb{X}_D \\ T_\mu = T}}}_{\text{replace this by an integral}} \varphi(h^{-1}\mu),\ f_1 \otimes f_2(h) \rangle \, dh$$

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$$\mu_0 \in \{\mu \in \mathbb{X}_D : T_\mu = T\} \circlearrowleft \underset{SO(D)_{\mathbb{Q}}}{\overset{H'_{\mathbb{Q}}}{\parallel}}$$ acts transitively

$$\text{Kernel} = \underset{SO(2)}{\overset{SO(\mu^\perp)}{\wr\parallel}}. \qquad L \xrightarrow{\mu} D$$

$$D = L \oplus \underset{SO(\mu^\perp)}{\overset{L^\perp}{\circlearrowleft}}$$

A_0

$$a(T)(1) = \int_{h \in H'(\mathbb{Q})\backslash H'(\mathbb{A})} \int_{SO(\mu^\perp)_{\mathbb{Q}}\backslash H'(\mathbb{Q}) \ni \gamma} \langle \varphi(h^{-1}\gamma^{-1}\cancel{\mu_0}), f_1 \otimes f_2(h) \rangle \, d\gamma \, dh.$$

$$= \int_{SO(\mu^\perp)_{\mathbb{Q}}\backslash H'(\mathbb{A})} \langle \varphi(\tilde{h}^{-1}\mu_0), f_1 \otimes f_2(\tilde{h}) \rangle \, d\tilde{h}$$

$$= \int_{SO(\mu^\perp)_{\mathbb{A}}\backslash H'(\mathbb{A})} \int_{SO(\mu^\perp)_{\mathbb{Q}}\backslash SO(\mu^\perp)_{\mathbb{A}}} \langle \varphi(h^{-1}t^{-1}\mu_0), f_1 \otimes f_2(th) \rangle \, dt \, dh$$

$$= \int_{SO(\mu^\perp)_{\mathbb{A}}\backslash H'(\mathbb{A})} \int_{SO(\mu^\perp)_{\mathbb{Q}}\backslash SO(\mu^\perp)_{\mathbb{A}}} \langle \varphi(h^{-1}\mu_0), f_1 \otimes f_2(th) \rangle \, dt \, dh$$

$\uparrow$

Use def.

- integrand is invariant under $N(\mathcal{D})_f = \{ \underset{(a,b)}{\overset{h}{\parallel}} \in H'(\mathbb{A}_f) \mid h^{-1}\mathcal{D}_f = \mathcal{D}_f \}$ $\subseteq$
- right inv. $H'(\mathbb{R})$

$$= (*) \int_{\underbrace{SO(\mu^{\perp})_f \backslash H'(\mathbb{A}_f) // N(\mathcal{D}_f)}_{\text{finite set}}} \int \left\langle \sum P_i^{k}(\mu_0) \otimes v_i^{k} e^{+2\pi \mathrm{Tr}(T)}, f_1 \otimes f_2(th) \right\rangle dt\,dh$$

$= \ldots$ ran out of time to finish.