Keaton

Introduction to modular symbols

§1 Modular forms:

Let $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$.

Let $\Gamma_0(N) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod N \} \leq SL_2(\mathbb{Z})$.

$$\Gamma_0(N) \circlearrowright \mathfrak{H}^* \quad \text{via} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Let $f : \mathfrak{H}^* \to \mathbb{C}$ be a holomorphic function s.t.

$$f(g\tau) = (c\tau + d)^k f(\tau) \qquad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

Then f is a modular form of weight k and level N.

$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(N)$

$f(\tau + 1) = f(\tau) \Rightarrow f$ has a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}.$$

If $a_0 = 0$ we say f is a cusp form.

$M_k(\Gamma_0(N)) = M_k(N) =$ complex vector space of modular forms of wt k and level $\Gamma_0(N)$.

The subspace of cusp forms is denoted $S_k(\Gamma_0(N))$.

$M_k(N)$ comes equipped with linear operators

$$T_n : M_k(N) \to M_k(N) \quad ; \quad n \in \mathbb{N}.$$
$$S_k(N) \to S_k(N)$$

We can find a basis for $M_k(N)$ consisting of

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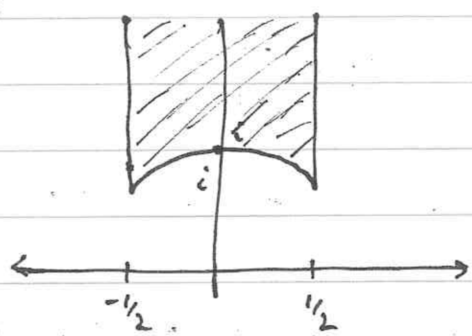
~~eigenforms~~ simultaneous eigenvectors for T_n, $(n,N)=1$.

<u>Goal:</u> Compute elements of $M_2(N)$.

§2 Modular Symbols:

Let $\Gamma \leq SL_2(\mathbb{Z})$.
Then
$$X_\Gamma := \Gamma \backslash \mathfrak{H}^*$$
is called a modular curve.
If $\Gamma = \Gamma_0(N)$, then $X_0(N) := X_\Gamma$.

<u>Example:</u>



$X_0(1)$

Let $M_2(\Gamma)^\vee = \mathrm{Hom}_{\mathbb{C}}(M_2(\Gamma), \mathbb{C})$.

Let $\alpha, \beta \in \mathfrak{H}^*$. Let $\{\alpha, \beta\}$ be a path from α to β in $\mathfrak{H}^*$.
Let $\{\alpha, \beta\}_\Gamma$ be the image of $\{\alpha, \beta\}$ in X_Γ.

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$\{\alpha,\beta\}_\Gamma$ determines an element of $M_2(\Gamma)^\vee$ via

$$f \longmapsto \int_\alpha^\beta 2\pi i\, f(z)\, dz.$$

We denote this map by $\{\alpha,\beta\}_\Gamma$ as well, it should be clear from context. These are <u>modular symbols</u>.

<u>Properties of modular symbols:</u>

1) $\{\alpha,\alpha\}_\Gamma = 0$
2) $\{\alpha,\beta\}_\Gamma + \{\beta,\alpha\}_\Gamma = 0$
3) $\{\alpha,\beta\}_\Gamma + \{\beta,\gamma\}_\Gamma + \{\gamma,\alpha\}_\Gamma = 0$
4) $\{g\alpha, g\beta\}_\Gamma = \{\alpha,\beta\}_\Gamma \quad \forall g\in\Gamma$
5) $\{\alpha, g\alpha\}_\Gamma = \{\beta, g\beta\}_\Gamma \quad \forall g \in \Gamma$
6) $\{\alpha, g_1 g_2\alpha\}_\Gamma = \{\alpha, g_1\alpha\}_\Gamma + \{\alpha, g_2\alpha\}_\Gamma \quad \forall g_1, g_2 \in\Gamma$

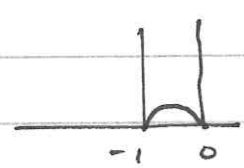
use $f(z)dz$ is invariant under Γ.

Triangulating $\mathfrak{h}^*$ (simplicial complex):

vertices: $\mathbb{P}^1(\mathbb{Q})$

edges: $\{\frac{a}{b}, \frac{c}{d}\} = \{g(0), g(\infty)\} \qquad g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$

triangles: $SL_2(\mathbb{Z})$ orbits of edges given by



$\{0,\infty\}$, $\{ST(0), ST(\infty)\}$

$\{(ST)^2(0), (ST)^2(\infty)\} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$

Replace $\{\alpha,\beta\}$ by $\{\alpha,\beta\}_\Gamma$ to obtain triangulation of X_Γ.

Notation: $(g) := \{g(0), g(\infty)\}_\Gamma \qquad g\in SL_2(\mathbb{Z})$

Relations: 1) $(g) = (g'g) \qquad g'\in\Gamma$

2) $(g) + (gS) = 0$

3) $(g) + (gST) + (g(ST)^2) = 0.$

Define: $C(\Gamma) := \mathbb{Z}[\Gamma \backslash SL_2(\mathbb{Z})]$

$B(\Gamma) := \langle \text{relation (2)}, \text{relation (3)} \rangle_{\mathbb{Z}}$

$Z(\Gamma) := \ker(\delta : C(\Gamma) \longrightarrow \mathbb{Z}[\Gamma \backslash \mathbb{P}^1(\mathbb{Q})])$

$(g) \longmapsto [g(\infty)]_\Gamma - [g(0)]_\Gamma$

rank $2g_\Gamma + g_0$ $\mathbb{Z}$-module $\{ C(\Gamma)/B(\Gamma) \xrightarrow{\sim} M_2(\Gamma)^\vee$

g_Γ = genus,

g_0 = # of cusps.

rank $2g_\Gamma$ $\mathbb{Z}$-module $\{ Z(\Gamma)/B(\Gamma) \longrightarrow S_2(\Gamma)^\vee.$

Set $\Gamma = \Gamma_0(N)$.

<u>Define</u>: a Manin-symbol is an element of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ where $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) := \{ (x,y) \in (\mathbb{Z}/N\mathbb{Z})^2 : \gcd(x,y,N) = 1 \} / (\mathbb{Z}/N\mathbb{Z})^\times.$

<u>Prop</u>: There is a bijection $\Gamma_0(N) \backslash SL_2(\mathbb{Z}) \longrightarrow \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (c:d).$$

We may also view $(c:d)$ as a modular symbol via

$$(c:d) \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \longmapsto (g).$$

We have an action

$$SL_2(\mathbb{Z}) \circlearrowright \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$$

$$(x:y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+cy : bx+dy).$$

In particular,

$$(x:y)S = (y:-x)$$
$$(x:y)ST = (y:y-x).$$

Our relations become:

2) $(c:d) + (-d:c) = 0$

3) $(c:d) + (d:d-c) + (d-c;-c) = 0.$

Our δ map becomes

$$\delta : (c:d) \longmapsto [a/c]_\Gamma - [b/d]_\Gamma.$$

$C(N) := \mathbb{Z}[\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})]$

$B(N) := \langle \text{relation (2), relation (3)} \rangle_\mathbb{Z}$

$Z(N) := \ker \delta.$

$$C(N)/_{B(N)} \longleftrightarrow M_2(N)^\vee$$

$$\left(C(N)/_{B(N)}\right)^\vee := \left\{ \begin{array}{l} \lambda : \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z} \\ \lambda((c:d)) + \lambda((-d:c)) = 0 \\ \lambda((c:d)) + \lambda((d:d-c)) + \lambda((d-c:c)) = 0 \end{array} \right\}$$

§3 Graph theoretic view:

Vertices : $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$

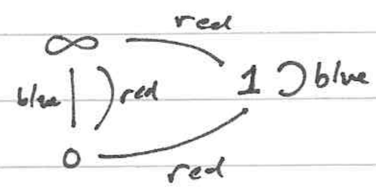
place a blue edge if $p = qS$

red edge if $p = qST$ or $p = q(ST)^2$

Example: : $N = 2$

$\infty = [1:0]$

$a = [a:1]$, $a \in \mathbb{Z}/N\mathbb{Z}$



Label our vertices from $\mathbb{Z}$.

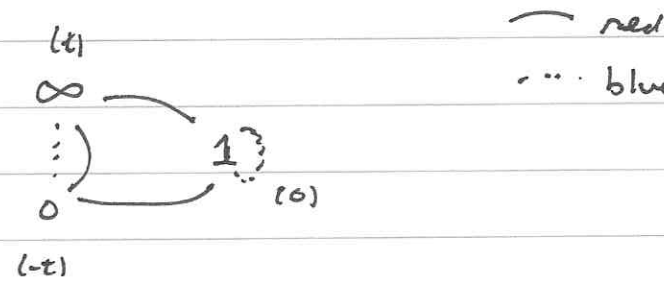
1) Labels of two vertices connected by a blue edge sum to 0

2) Labels of three vertices ~~by~~ connected by a red edge sum to 0

Let $\mathcal{L}(N) = \{ \lambda : \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z} : \lambda(p) + \lambda(pS) = 0$
$\lambda(p) + \lambda(pST) + \lambda(p(ST)^2) = 0 \}$.

Observe

$$\mathcal{L}(N) = \left(C(N)/B(N) \right)^\vee .$$



Hecke operators act on $\mathcal{L}(N)$ via

$$(T_\ell \lambda)(x:y) = \sum_{\substack{a > b \geq 0 \\ c > d \geq 0 \\ \gcd(ax+cy,\, bx+dy,\, N) = 1 \\ ad - bc = \ell}} \lambda\left((x:y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

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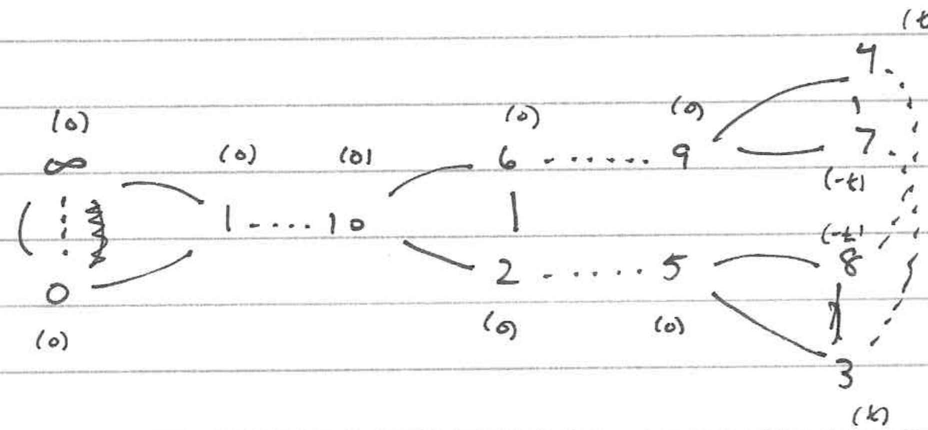
<u>Example:</u> $N = 11$



$\dim_{\mathbb{C}} S_2(\Gamma_0(11)) = 2$

$(T_2 \lambda)(4) =$

$\lambda([4:1]\begin{pmatrix}1&0\\0&2\end{pmatrix}) + \lambda([4:1]\begin{pmatrix}2&0\\0&1\end{pmatrix})$
$+ \lambda([4:1]\begin{pmatrix}2&1\\0&1\end{pmatrix}) + \lambda([4:1]\begin{pmatrix}1&0\\1&2\end{pmatrix})$

$= \lambda(2) + \lambda(8) + \lambda(6) + \lambda(8)$

$= 0 - a + 0 - a = \underbrace{-2a}$

eigenvalue at 2.

This graph ~~of~~ corresponds to

$$f(\tau) = q - 2q^2 - q^3 + O(q^4) \in S_2(11).$$