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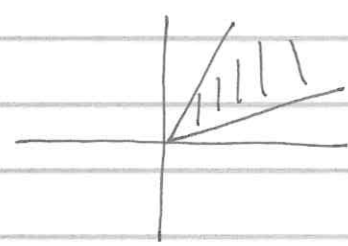
Conic Theta Functions:

joint work w/ A. Folsom and S. Robins (2011/12)

Plan: ① Solid angles and conic theta functions
② Non-modularity
③ Modularity

① Recall:

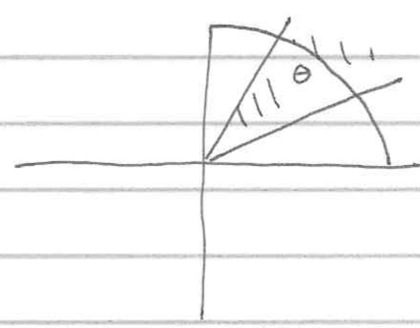
Def: Let $\omega_1, \ldots, \omega_d \in \mathbb{R}^d$ be a basis. Then $K = \{ \sum_{j=1}^{d} \lambda_j \omega_j : \lambda_j \geq 0 \ \forall j \}$. This is called a polyhedral cone with edges $\omega_1, \ldots, \omega_d$



Ex: Pos orthant $K_0 = \mathbb{R}^d_{\geq 0}$.

Def: One defines the solid angle ω_K at the vertex of K (= origin) by

$$\omega_K := \frac{\mathrm{vol}(K \cap S^{d-1})}{\mathrm{vol}(S^{d-1})}$$



Remark: i) One has $0 < \omega_K < 1$.

ii) If $d=2$, then $\omega_K = \theta/2\pi$ where θ is the usual 2-dim. angle measured in radians.

Lemma 1: $$\omega_K = \int_K e^{-\pi \|x\|^2}\,dx$$

Proof: Use polar coordinates, i.e., $x = rs$ with $r = \|x\|$ and $s \in S^{d-1}$. Then $dx = r^{d-1}\,dr\,ds$. Therefore,

$$\frac{\int_K e^{-\pi\|x\|^2}\,dx}{\underbrace{\int_{\mathbb{R}^d} e^{-\pi\|x\|^2}\,dx}_{=1}} = \frac{\int_0^\infty r^{d-1} e^{-\pi r^2}\,dr \int_{K\cap S^{d-1}} ds}{\int_0^\infty r^{d-1} e^{-\pi r^2}\,dr \int_{S^{d-1}} ds}$$

$$= \frac{\mathrm{vol}(K\cap S^{d-1})}{\mathrm{vol}(S^{d-1})} = \omega_K. \qquad \blacksquare$$

Def: Let $L \subseteq \mathbb{R}^d$ be a (full) lattice, K polyhedral cone. Then

$$\Phi_{K,L}(z) = \sum_{m\in L\cap K} e^{\pi i \|m\|^2 z} \qquad (z \in \mathfrak{H})$$

is called a conic theta function.

Lemma 2: Write $L = A\mathbb{Z}^d$ where $A \in GL_d(\mathbb{R})$. Then

$$\lim_{t\to 0} t^{d/2}\,\Phi_{K,L}(it) = \frac{\omega_K}{|\det A|}.$$

Proof: Use Riemann sums: $f(x) = e^{-\pi \|x\|^2}$ $(x \in \mathbb{R}^d)$

$$\omega_K = \int_K f(x)\,dx = \lim_{\Delta h \to 0} \sum_{m \in K \cap \mathbb{Z}^d} (\Delta h)^d f(m \cdot \Delta h),$$

where $(\Delta h)^d$ is the d-dimensional volume of a small cube of side length Δh, intersecting in sets of measure zero only and covering K. Choose $\Delta h = t^{1/2}$.

Then

$$\omega_K = \lim_{t \to 0} \sum_{m \in K \cap \mathbb{Z}^d} t^{d/2} f(t^{1/2} m)$$

$$= \lim_{t \to 0} t^{d/2} \Phi_{K,L}(it). \blacksquare$$

② Non-modularity

Recall: $L \subseteq \mathbb{R}^d$ lattice, write $L = A\mathbb{Z}^d$ with $A \in GL_d(\mathbb{R})$, $B = {}^tA A$. We have a quadratic form

$$Q(x) = {}^t x B x \qquad (x \in \mathbb{R}^d)$$

the associated pos. def quadratic form, assume B even integral. Then

$$\vartheta_L(z) = \sum_{m \in L} e^{\pi i \|m\|^2 z} \qquad (z \in \mathfrak{H})$$

$$= \sum_{m \in \mathbb{Z}^d} e^{\pi i Q(m) z}$$

$$= 1 + \sum_{l=1}^{\infty} c(l) q^l \qquad (q = e^{2\pi i z})$$

(where $c(\ell) = \#\{n \in \mathbb{Z}^d : \frac{1}{2}Q(\ell) = n\}$.)

is a modular form of weight $d/2$ on $\Gamma_0(N)$ where N = level of Q (smallest pos. integer M s.t. $M B^{-1}$ is even integral.), and where

$$\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) : N \mid c \}.$$

This essentially means that

$$\vartheta_L\left(\frac{az+b}{cz+d}\right) = (*)\,(cz+d)^{d/2}\,\vartheta_L(z) \quad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N).$$

In general, one would not expect that $\Phi_{k,L}(z)$ are modular!

<u>Question:</u> How to prove this? How "non-modular" are they?

<u>Theorem 1:</u> 1) Suppose that $\Phi_{k,L}$ is a modular form of integral or half-integral weight k on $\Gamma_0(M)$ for some $M \in \mathbb{N}$. Then necessarily one must have $k = d/2$.

2) Suppose that $\frac{\omega_k}{|\det A|} \notin \mathbb{Q}$. Then $\Phi_{k,L}$ is <u>NOT</u> a modular form of weight $d/2$.

<u>Proof:</u> 1) Suppose $\Phi_{k,L}$ is modular of weight k. Then it is holo. at the cusp 0, so in particular the value

$$\lim_{t\to 0} t^k\, \Phi_{k,L}(it) = b_0$$

exists and is finite. By Lemma 2,

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$$\lim_{t \to 0} t^{d/2}\, \Phi_{K,L}(it) = \frac{\omega_K}{|\det A|} > 0.$$

i) if $k < d/2 \Rightarrow b_0 = \infty$ #

2) if $k > d/2 \Rightarrow b_0 = 0$. Let $a(n) = n^{th}$ Fourier coefficient of $\Phi_{K,L}$. $(n \geq 0)$. So in particular $a(0) = 1$. Let $D(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ $(Re(s) >> 1)$. the Hecke L-series attached to $\Phi_{K,L}$. By Hecke we know $D(s)$ has meromorphic continuation to $\mathbb{C}$ and is holomorphic except for a possible simple pole at $s = b_0 = 0$. Thus, $D(s)$ is holo. on $\mathbb{C}$. By Landau's theorem, since $a(n) \geq 0$ $\forall n \geq 1$, $D(s)$ must converge $\forall s \in \mathbb{C}$, so

$$a(n) = O(n^c) \quad \forall c \in \mathbb{R}.$$

By Schmoll-K. (2010) if $b(n)$ $(n \geq 0)$ are the coeff. of a modular form of wt k and

$$b(n) = O_f(n^{\frac{k-1}{2}+\varepsilon}) \quad (\varepsilon > 0)$$

$\forall n \geq 1 \Rightarrow g$ is a cusp form. Apply this to see $\Phi_{K,L}$ is a cusp form. # since $a(0) = 1 \neq 0$.

2) Use the q-exp. principle (Deligne-Rapoport): $\Phi_{K,L}$ has Fourier coeffs in $\mathbb{Q}$. If it were modular of wt $d/2$, then the Fourier coeff at the cusp 0 is in $\mathbb{Q}$ by this principle. # to $\frac{\omega_K}{|\det A|} \notin \mathbb{Q}$ by Lemma 2. ∎

Theorem 2: Suppose $d = 2$ and K is an integral cone, i.e. $\omega_1, \omega_2 \in \mathbb{Z}^2$. Then $\Phi_{K,\mathbb{Z}^2}$ is not modular of wt 1.

Proof: In most cases, one can use the q-exp principle:

$\omega_K = \frac{\Theta}{2\pi}$, suppose ~~and~~ then $\Theta = 2\pi\, c/d$ with

$c, d \in \mathbb{Z}$, $d \neq 0$. Also, $\cos\Theta = \frac{\langle w_1, w_2\rangle}{\|w_1\|\,\|w_2\|} = \frac{n}{\sqrt{m}}$

(for $n, m \in \mathbb{Z}$ $m \geq 0$ b/c $w_1, w_2 \in \mathbb{Z}^2$)

$\Rightarrow$

$$\cos\left(2\pi \frac{c}{d}\right) = \frac{n}{\sqrt{m}} \quad \text{is rational or quadratic.}$$

$\downarrow$

cyclotomic of $\frac{\varphi(d)}{2}$ #. ■

3) Modularity:

Thm 3: Let K be the Weyl chamber of a finite Weyl reflection group attached to one of the root systems A_n, B_n, C_n, D_n. Let L_{root} be the corresponding root lattice. Then $\Phi_{K, L_{root}}$ lies in the graded ring of modular forms.

Proof: "Morally"

" $\Phi_{K,L} \cdot |W| = \vartheta_L$ ".

induction on d: