Generalized Heegner cycles and congruences of p-adic L-functions

Motivation: Let $E/\mathbb{Q}$ be an elliptic curve. BSD predicts $\operatorname{rk} E(\mathbb{Q}) = \operatorname{ord}_{s=1} L(E/\mathbb{Q}, s)$. When $\omega(E/\mathbb{Q}) = 1$, generically expect alg rk = an rk = 0. When $\omega(E/\mathbb{Q}) = -1$, generically expect alg rk = an rk = 1.

Quadratic Twists: Given a fundamental disc. D, $\exists!$ (up to isom $/\mathbb{Q}$) $E^D/\mathbb{Q}$ s.t. $E/\mathbb{Q}(\sqrt{D}) \simeq E^D/\mathbb{Q}(\sqrt{D})$

$E: y^2 = x^3 + ax + b \qquad E^D \quad Dy^2 = x^3 + ax + b$

If $(D, N) = 1$ where $N = \operatorname{cond}(E)$, then $\omega(E^D/\mathbb{Q}) = \left(\frac{D}{-N}\right)\omega(E/\mathbb{Q})$.

Goldfeld's Conjecture: (1979) Let $N_r(E, x) = \#\left\{ \text{fund. disc. } D : |D| < x, \ \operatorname{ord}_{s=1} L(E^D, s) = r \right\}$ for $r = 0, 1$.

As $x \to \infty$ $\qquad N_r(E, x) \sim \frac{1}{2} \sum_{|D| < x} 1$

Assuming BSD we expect half of quadratic twists have alg. rank $= r$ for $r = 0, 1$.

Remark: In the function field setting there is an analogous conjecture. Katz and Sarnak studied symmetry groups and monodromy of vars $/\mathbb{F}_q$ and along with random matrix calculation of Keating & Smith formulated and provided evidence for a more precise version of Goldfeld's conjecture.

What is known? (over number fields? $\mathbb{Q}$, or totally real.

very little! (Need to adjust if not totally real)

(WGC): Weak version of the conjecture: As $X \to \infty$, $N_r(E, X) \gg X$ for $r = 0, 1$.

Iwaniec gave a proof of the weak version assuming GRH.

Ono - Skinner : $N_0(E, X) \gg \frac{X}{\log X}$ using Waldspurger, and Friedberg-Hoffstein.

Ono : $N_0(E, X) \gg \frac{X}{\log^{1-\alpha} X}$

Perelli - Pomykala : $N_1(E, X) \gg X^{1-\varepsilon}$ for $\varepsilon > 0$, depends on E.

First examples of WGC :

$r = 0$ 14a6 James

$r = 1$ 14a1 Vatsal $/\mathbb{Q}(\sqrt{D})$

Vatsal constructed first infinite family $E/\mathbb{Q}$ semi-stable,

$$E[3](\mathbb{Q}) \neq 0 \Rightarrow N_0(E, X) \gg X.$$

h_D^2

Pf idea: uses $E[3](\mathbb{Q}) \neq 0 \Rightarrow L(E^D, 1)^{alg} \equiv \left(\begin{smallmatrix}\text{Euler}\\ \text{factors}\end{smallmatrix}\right) L\left(\left(\frac{D}{\cdot}\right), 0\right)^2 \pmod 3$

of $L\left(\left(\frac{D}{\cdot}\right), 0\right)$ at $\ell \mid N$

Assuming cong. conditions on D mod N and Davenport - Heilbronn.

Thm (K'19): Let $E/\mathbb{Q}$, N be as above, $p \nmid N$ $p > 2$, and

$E[p]$ reducible $G_{\mathbb{Q}}$ ($\Leftrightarrow E[p]^{ss} \simeq \mathbb{F}_p(\psi) \oplus \mathbb{F}_p(\psi^{-1}\omega)$)

prim. Dir. char. $\psi : G_{\mathbb{Q}} \to \mu_{p-1}$, ω Teichmuller char.

Assume:

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(1) $\psi(p) \neq 1$

(2) E has no primes of split mult. red.

(3) $\ell \mid N$ add. reduction $\Rightarrow$ either $\psi(\ell) \neq 1$ and $\ell \not\equiv -1 \pmod{p}$

or $\psi(\ell) = 0$.

and let K be an imag. quad. field s.t.

(1) p splits in K

(2) K satisfies Heegner hypothesis ($\ell \mid N \Rightarrow \ell$ split in K)

(3) $p \nmid B_{1,\psi_0^{-1}\varepsilon_K} B_{1,\psi_0\omega^{-1}}$ where

$$\psi_0 = \begin{cases} \psi, & \psi(-1) = 1 \\ \psi\varepsilon_K & \psi(-1) = -1 \end{cases}$$

$\varepsilon_K =$ quad char. assoc. to K

Then $\operatorname{rk} E(K) = \operatorname{ord}_{s=1} L(E/K, s) = 1$.

Remark: $p=3$, $\psi =$ quad. , Bernoulli numbers in statement are quadratic class numbers can again apply Davenport - Heilbronn to get:

Cor: Suppose $E[3]$ reducible

(1) $N_0^{(\text{alg}/\text{an})}(E,X) + N_1^{(\text{alg}/\text{an})}(E,X) \gg X$

(2) If E semi-stable, then

$$N_0^{(\text{alg}/\text{an})}(E,X) \gg X \quad \text{and} \quad N_1^{(\text{alg}/\text{an})}(E,X) \gg X.$$

In particular, WGC holds for E.

Idea of proof (of Thm): If $E[p]$ reducible,

$$E[p]^{ss} \simeq \mathbb{F}_p(\psi) \oplus \mathbb{F}_p(\psi^{-1}\omega)$$

$f \longleftrightarrow E \qquad \ell \nmid N$

normalized newform

$\Rightarrow \quad a_\ell \equiv \psi(\ell) + \psi^{-1}(\ell)\ell \pmod{p}$

$a_\ell = \pm 1, 0$ when $\ell \mid N$.

Assume $a_\ell \equiv \psi(\ell) \pmod{p}$ or $\psi^{-1}(\ell)\ell \pmod{p}$.

$$\Rightarrow \quad f(q) \equiv E_{2,\psi}^{(N)}(q) \pmod{p}$$

$$\ell \mid N \quad E_{2,\psi}^{(\ell)}(q) = E_{2,\psi}(q) - \psi^{-1}(\ell)\,\ell\, E_{2,\psi}(q^\ell).$$

$\theta = q\frac{d}{dq} \qquad \theta^j\left(\sum a_n q^n\right) = \sum n^j a_n q^n$

$$\Rightarrow \quad \theta^j f(q) \equiv \theta^j E_{2,\psi}^{(N)}(q) \pmod{p}$$

q-exp. prin

$$\Rightarrow \quad \theta^j f \equiv \theta^j E_{2,\psi}^{(N)} \text{ on ordinary locus.}$$

p splits in K $\Rightarrow$ $A/_{\mathbb{C}_p}$ CM curve, $\mathrm{End}_H(A) = \mathcal{O}_K$ is ordinary by Deuring.

central critical anti cyclo ch. $/K$

$$\Rightarrow \sum_{\substack{\text{corresp to} \\ \sigma \in Cl(\mathcal{O}_K)}} \theta^j f(\sigma\cdot\tau)\,\chi^{-1}(\sigma) \equiv \sum_{\sigma \in Cl(\mathcal{O}_K)} \theta^j E_{2,\psi}^{(N)}(\sigma\tau)\,\chi^{-1}(\sigma) \pmod{p} \equiv \underbrace{\Xi_\chi(N)}_{\text{Euler factor}} \sum_\sigma E_{2,\psi}^{(N)}(\sigma\tau)\,\chi^{-1}(\sigma).$$

$$\Rightarrow \theta^j f^{(p)} \equiv \theta^j E_{2,\psi}^{(N)} \quad \text{for } j \in \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$$

$$\Rightarrow \mathcal{L}_p^{BDP}(f, \chi^{-1}) \equiv \underbrace{\Xi_{\chi^{-1}}(N)^2}_{\text{Euler factor}} \mathcal{L}_p^{Katz}(\psi \cdot N_K, \chi^{-1}, 0) \pmod{p}$$

BDP $\Rightarrow$ p-adic Waldspurger thm at $\chi = N_K$

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$$\left(\frac{1-a_p+p}{p}\right) \log_E^2 P_E(K) \equiv \underbrace{\Xi_{N_K^{-1}}(N)\, L_p^{KL}(\psi^{-1}\varepsilon_\chi\, \omega, 0)}\; L_p^{KL}(\psi,1) \pmod p$$

$\not\equiv 0 \pmod p$ guaranteed by our assumptions.

<u>Example:</u> Let $E : y^2 + y = x^3 + x^2 - 9x - 15$ 19a1.

$E[3](\mathbb{Q}) \neq 0$, $N = 19$ split mult. red.

Let $D = 41$, $K = \mathbb{Q}(\sqrt{-2})$.

- 3 and 19 split in K
- $3 \nmid h_{\mathbb{Q}(\sqrt{123})}\, h_{\mathbb{Q}(\sqrt{-82})}$.

$\Rightarrow$ $\operatorname{rk} E^{41}(K) = \operatorname{ord}_{s=1} L(E^{41}/K, s) = 1$

$\operatorname{rk} E^{41}(\mathbb{Q}) = \operatorname{ord}_{s=1} L(E^{41}/\mathbb{Q}, s) = 0$

$\operatorname{rk} E^{-82}(\mathbb{Q}) = \operatorname{ord}_{s=1} L(E^{-82}/\mathbb{Q}, s) = 1$.

Cor $\frac{323}{10240} \approx 3.16\%$ of real quad. twists have rk 1

$\frac{323}{3584} \approx 9.04\%$ of imag. quad. twists have rk 0.

We now discuss generalizations.

<u>Motivation:</u> Beilinson-Bloch conjecture.

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$X/_F$ nonsingular variety $/_F$, F = number field.

$CH^j(X)$ = Chow group codim j cycles

~~$d(X,Y)$~~ $CH^{\dim Y + d}(X \times Y)$

$Corr^d(X, Y) :=$

"motive" (X, e, m) e idempotent in $Corr^0(X, X)$

$m \in \mathbb{Z}$.

Let $W_r = \mathcal{E}^r$ (canonical desingularization to smooth fibers at cusps)

$$\begin{array}{ccc} \mathcal{E} & \text{univ. elliptic curve} & \mathcal{E} \\ \downarrow & & \downarrow \\ Y_1(N) & & X_1(N) \end{array}$$

Let $f \in S_k(\Gamma_1(N))$ wt $k \geq 2$, $k = r+2$.

Scholl constructed a projector ε_f s.t.

$$\varepsilon_f(H^{r+1}_{et}(W_{r,\bar{\mathbb{Q}}}, \mathcal{O}_\lambda) \otimes_{\mathbb{Q}} E_f) \simeq V_\lambda(f)$$

λ-adic Gal. rep $\longleftrightarrow$ f.

- $M_f = (W_r, \varepsilon_f, 0)/_{\mathbb{Q}}$ coeffs. in E_f.

- Fix A CM curve $End_K(A) = \mathcal{O}_K$ Let χ be a Hecke char. $/K$ of inf. type $(r+j, r-j)$

- Have motive $(A^r, \varepsilon_\chi, 0)$

Choose abelian F/K, $\chi \circ N_{F/K} = \chi_A^{r-j} \bar{\chi}_A^{\,j}$, χ_A inf type $(1,0)$ $\leftrightarrow$ A of CM

. Deninger constructs projector $(\chi_{/F} = \chi \circ N_{F/K})$

$$\varepsilon_{\chi/F} H^r_{et}(A^r_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) \simeq (\kappa \otimes \mathbb{Q}_\ell)(\chi_{/F})$$

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- $M_{\chi/F} := (A^r{}_{/F}, \varepsilon_{\chi/F}, 0)$

- Define Rankin-Selberg motive

$$M_{(f,\chi^{-1})/F} = M_{f/F} \otimes M_{\chi^{-1}/F}$$

$$= (\underbrace{W_{r/F} \times A^r{}_{/F}}_{\text{generalized Kuga-Sato var}}, \underbrace{\varepsilon_{(f,\chi^{-1})/F}}_{\varepsilon_{f/F}\,\varepsilon_{\chi^{-1}/F}}, 0).$$

<u>Beilinson-Bloch Conj:</u>

$$\dim_{E_{f,\chi}} \varepsilon_{(f,\chi^{-1})/F} CH^{r+1}(X_r)_{0,E_{f,\chi}}(F)$$

$$= \operatorname{ord}_{s=r+1} L(\underbrace{H^{2r+1}_{et}}_{\text{any } \ell \nmid N \text{-adic realization}}(M_{(f,\chi^{-1})/F}, s))$$

$$\underset{F/k \text{ abelian}}{=} \operatorname{ord}_{s=s_0} \prod_{\chi_0 : Gal(F/k) \to \mathbb{C}^\times} L(\underbrace{\pi_f \times \pi_{\chi^{-1}\chi_0}}_{\text{auto reps} \leftrightarrow (f,\chi) \text{ unitary norm.}}, s)$$

Heegner hyp. $\Rightarrow L(\pi_f \times \pi_{\chi^{-1}}, 1/2) = 0$.

Thus, $\dim_{E_{f,\chi}} \varepsilon_{(f,\chi)} CH^{r+1}(\ldots) \geq 1.$

Candidate for nontrivial cycle class is generalized Heegner cycle.

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- $\varphi : A \to A'$, A' - CM curve
- $\Delta_\varphi = \mathcal{E}_X \Gamma_\varphi^r \subseteq A^r \times (A')^r \subseteq A^r \times W_r = X_r.$

- BDP relates p-adic Abel-Jacobi image of a gen. Heegner cycle to a special value of their p-adic L-function.
- $\bar{\rho}_f$ reducible residual Galois rep. , $\simeq k_\lambda(\psi_1) \oplus k_\lambda(\psi_2 \omega)$

$$\Rightarrow \mathcal{L}_p^{BDP}(f, \chi^{-1}) \equiv (*) \frac{\Xi_{\chi f}(N)}{(2\pi i)^{\text{some power I can't read}}} L_1^{\text{Katz}}(\psi_1/_K \chi^{-1}, 0) \pmod{\lambda}$$

mod λ criterion, non p-divisibility of certain Bernoulli number

to show $AJ(\mathcal{E}_{(f,\chi)/F} \Delta) \neq 0 \Rightarrow \mathcal{E}_{(f,\chi)/F} \Delta \neq 0.$

<u>Non-Eisenstein Case</u> (joint w/ ~~Chao~~ Chao Li)

$\bar{\rho}_f$ irred.

Assume $E/\mathbb{Q}$, $E[2]$ irred. Galois rep.

Let K be an imag. quad. field.

<u>Def</u>: Let S be the set of primes $\ell \nmid N$ s.t.

1) ℓ splits in K (density 1/2)

2) Frob_ℓ has order 3 in Galois group $\text{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \cong S_3$

(density 1/3)

Suppose ~~D~~ d = squarefree product $\ell \in S$.

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Thm: Suppose E has good supersingular reduction at 2

and
$$\frac{\log_E P_E(K)}{2} \not\equiv 0 \pmod 2.$$

Then
$$\frac{\log_{E^d} P_{E^d}(K)}{2} \not\equiv 0 \pmod 2$$

$\Rightarrow$ $P_{E^d}(K)$ non-torsion, $\mathrm{rk}\, E^d(K) = 1$.

Pf: Establish congruence between
$$\log_E P_E(K) \equiv \Xi(N) \log_{E^d} P_{E^d}(K) \pmod 2$$

which follows from $f_E \equiv f_{E^d} \pmod 2$.

Example: $E = 37a1$: $y^2 + y = x^3 - x$ s.s good red. at 2.

Choose $K = \mathbb{Q}(\sqrt{-7})$, 2, 37 split in K

$P = P_f(K) = (0,0)$, $5P = (\frac{1}{4}, -\frac{5}{8}) \xrightarrow{\text{reduces}} \infty \in \tilde{E}(\mathbb{F}_2)$.

$t = -x/y$

norm. inv. diff. $\omega_E(t) = 1 + 2t^3 - 2t^4 + \dots$

$\log_E(t) = t + \frac{1}{2}t^4 - \frac{2}{5}t^5 + \frac{6}{7}t^7 + \dots$

$5P \leftrightarrow t = 2/5$

$\log_E(2/5) = \frac{2}{5} + \frac{1}{2}(\frac{2}{5})^4 + \dots \in 2\mathbb{Z}_2^\times$. Can check using SAGE

$\mathrm{rk}\, E^\ell(\mathbb{Q}(\sqrt{-7})) = 1$ for many primes $\ell \in S$.

Not true for general $\ell \notin S$.