On the integral kernel for a multiple Dirichlet series associated to Siegel cusp forms.

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Elliptic cusp forms:

$k > 0$, even.

$f \in S_{1,k}(SL_2(\mathbb{Z}))$, $\Gamma_1 = SL_2(\mathbb{Z})$.

$f(z) = \sum_{n \geq 1} c(n) e(nz)$; $e(x) = e^{2\pi i \operatorname{tr} X}$

Associated we have

$D(f, w) = \sum_{n \geq 1} c(n) n^{-w}$. , $w \in \mathbb{C}$. → Mellin transform
→ via Petersson inner product.

For $w \in \mathbb{C}$ fixed, $\begin{aligned} S_{1,k} &\longrightarrow \mathbb{C} \\ f &\longmapsto D(f, w) \end{aligned}$ is linear

Then $\exists\ \Omega_w \in S_{1,k}$ s.t.

$$\langle \Omega_w, f^* \rangle = * D(f, w)$$

powers of i, 2, π, gamma fctns, zeta fctns.

and $f^*(\tau) = \overline{f(-\bar{\tau})}$.

Would like to describe Ω_w more precisely.

Cohen, Kohnen, Diamantis-O'Sullivan have worked on this:

$$\Omega_w(\tau) \doteq \sum_{g \in \Gamma_1} \left(\frac{1}{\tau}\right)^w \Big|_k [g]$$

Remark:

$\Omega_w(\tau) = \sum_{n \geq 1} n^{w-1} P_{1,k,n}(\tau)$ where $P_{1,k,n}$ are Poincaré series

$$P_{1,k,n}(\tau) = \sum_{g \in \Gamma_{1,\infty} \backslash \Gamma_1} e(n\tau)|_k[g].$$

Siegel Cusp forms:

$$\mathfrak{h}_2 = \left\{ Z = \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \in Mat_2(\mathbb{C}) : {}^tZ = Z,\ Y > 0 \right\}.$$

$\Gamma_2 = Sp_2(\mathbb{Z})$ acts on $\mathfrak{h}_2$ where

$$\Gamma_2 = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Mat_4(\mathbb{Z}) : J[M] = J \right\}$$

$$J = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \quad \text{and} \quad A[B] = {}^tBAB.$$

The action is given by $Z \longmapsto MZ = (AZ+B)(CZ+D)^{-1}$.

Def: For $F : \mathfrak{h}_2 \to \mathbb{C}$ holomorphic, for $M \in \Gamma_2$ set

$$F|_k M(Z) = \det(CZ+D)^{-k} F(MZ).$$

Def: A Siegel modular form is a fctn $F : \mathfrak{h}_2 \to \mathbb{C}$ s.t. F is holomorphic on $\mathfrak{h}_2$ satisfying $F|_k[M] = F$ for all $M \in \Gamma_2$.

The set of all of them $M_{2,k}$ is a finite dimensional v.s.

Def: $\mathcal{P} = \{ Y \in Mat_2(\mathbb{R}) : {}^tY = Y,\ Y \text{ pos def} \}$.

$\mathcal{J} = \left\{ T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in \mathcal{P} : n, r, m \in \mathbb{Z} \right\}$.

Remark: $\mathfrak{h}_2 = \mathcal{S} + i\mathcal{P}$

symm matrices in $Mat_2(\mathbb{R})$.

$SL_2(\mathbb{R}) \curvearrowright \mathcal{P}$ via $Y \mapsto Y[g]$.

$\Gamma_1 \curvearrowright \mathcal{J}$

<u>Def:</u> A Siegel cusp form is $F \in M_{2,k}$ s.t.

$$F(Z) = \sum_{T \in \mathcal{J}} c(T)\, e(TZ)$$

We denote the set of Siegel cusp forms by $S_{2,k}$.

$S_{2,k}$ is a subspace of $M_{2,k}$ with a Peter. inner product

$$\langle F, G \rangle = \int_{\Gamma_2 \backslash \mathfrak{h}_2} F(Z)\, \overline{G(Z)}\, (\det Y)^{k/2}\, d\mu(Z).$$

<u>Example:</u>

$$P_{2,k,T}(Z) = \sum_{M \in \Gamma_{2,\infty} \backslash \Gamma_2} e(TZ)\big|_k [M]$$

where $\Gamma_{2,\infty} = \left\{ \begin{pmatrix} 1_2 & B \\ 0 & 1_2 \end{pmatrix} \,\middle|\, B \in \mathrm{Mat}_2(\mathbb{Z}),\ {}^tB = B \right\}$.

<u>Def:</u> $F \in S_{2,k}$, $w \in \mathbb{C}$, the Koecher-Maass Dirichlet series associated to F is

$$D(F, w) = \sum_{T \in \mathcal{J}/\Gamma_1} \frac{c(T)\,(\det T)^{-w}}{\xi_T}$$

where $\xi_T = \#\{ g \in \Gamma_1 : T[g] = T \}$ $\quad (c(T[g]) = c(T))$.

As before, for $\omega \in \mathbb{C}$ fixed, the function $F \mapsto D(F,\omega)$ is a linear function on $S_{2,k}$ so there exists $\Omega_\omega \in S_{2,k}$ s.t. $\langle \Omega_\omega, F^* \rangle = * \, D(F,\omega)$. We would like to describe Ω_ω in this case as well.

$$\Omega_\omega(Z) = \sum_{M \in *\backslash\Gamma_2} \left(\frac{1}{\det Z}\right)^\omega \Big|_k [M]$$ (Kohnen-Sengupta '02).

<u>Def:</u> A Maass waveform is a function
$$U : \Delta\mathcal{P} = \{ Y \in \mathcal{P} : \det Y = 1 \} \longrightarrow \mathbb{C}$$
s.t. U is real analytic, $U(Y[g]) = U(Y)$ for all $g \in \Gamma_1$,
$\Delta U = \lambda U$ where Δ = hyperbolic Laplacian.
λ is in $\left\{ \begin{array}{l} \text{discrete} \\ \text{cont.} \end{array} \right.$ spectrum. We only consider when λ is in the continuum spectrum. Then $\lambda = s(s-1)$ and
$$U(Y) = \mathcal{Z}(Y,s) = \sum_{\substack{(u,v) \in \mathbb{Z}^2 \\ (u,v) \neq (0,0)}} \left(Y\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \right)^{-s}.$$

<u>Def:</u> For $F \in S_{2,k}$, $\omega \in \mathbb{C}$,
$$D(F,s,\omega) = \sum_{T \in \mathcal{J}/\Gamma_1} \frac{c(T)}{\xi_T} \mathcal{Z}\left(\frac{T}{\sqrt{\det T}}, s\right) (\det T)^{-\omega}$$

Again, the function
$$F \longmapsto D(F,s,\omega)$$ is linear so there exists

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$\Omega_{w,s} \in S_{2,k}$ s.t.

$$\langle \Omega_{s,w}, F^* \rangle = * \, D(F, s, w).$$

<u>Problem:</u> $\Omega_{s,w} = ?$ properties? consequence for $D(F,s,w)$? consequences for F?

<u>Tools:</u> We need analogue of $Z \mapsto (\det Z)^{w}$

<u>Def:</u> s, w fixed $\quad p_{s,w} : \mathfrak{H}_2 \to \mathbb{C}$

$$Z \longmapsto \tau_1^{s} (\det Z)^{w}$$

$$\begin{pmatrix} \tau_1 & * \\ * & * \end{pmatrix}$$

It is an interesting function:

a) $\Gamma(s) = \int_0^\infty y^s e^{-y} \frac{dy}{y}$; $\Gamma_2(s,w) = \int_{\mathcal{P}} p_{s,w}(iY) e^{-\operatorname{tr} Y} d\mu(Y)$

$$= * \, \Gamma(s+w)\Gamma(w - 1/2)$$

b) $$\sum_{m \in \mathbb{Z}} (\tau + m)^{-w} = \frac{*}{\Gamma(w)} \sum_{n \geq 1} (n^{-1})^{1-w} e(n\tau).$$

$$\sum_{\substack{B \in \mathrm{Mat}_2(\mathbb{Z}) \\ {}^tB = B}} p_{-s,s-w}(Z+B) = \frac{*}{\Gamma_2(-s,w)} \sum_{T \in J} p_{-s,\, s+3/2-w}(iT^{-1}) e(TZ).$$

<u>Def:</u> For $Y \in \mathcal{P}$, $s, w \in \mathbb{C}$

$$E(Y,s,w) = \sum_{g \in \Gamma_{1,\infty} \backslash \Gamma_1} p_{-s,-w}(iY[g]).$$

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Remark: $D(F,s,w) = \zeta(2s) \sum_{T \in J/\Gamma_1} \frac{c(T)}{\xi_T} E(T, s, w - s/2).$

Results:

$\Omega : \mathbb{C}^2 \times \mathfrak{H}_2 \to \mathbb{C}$

Def: $\Omega_{s,w}(Z) = \sum_{M \in G \backslash \Gamma_2} P_{-s,-w}(Z)|_k [M]$

$G = \left\{ \begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix} : {}^tU \in \Gamma_{1,\infty} \right\} \leq \Gamma_2.$

Prop.: $k \geq 12$,

$$\Omega_{s,w}(Z) = \sum_{T \in J/\Gamma_1} \frac{1}{\xi_T} E(T, s, -s-w+3/2)\, P_{2,k,T}(Z).$$

Thm: $k \geq 12$

1) $\Omega_{s,w}$ has meron. cont. to $\mathbb{C}^2$ with possible singularities at (s,w) with $s = 0, 1/2, 1$

2) $\Omega_{s,w} = \Omega_{s,\, k-s-w} = -i\, e^{-\pi i s}\, \Omega_{1-s,\, s+w-1/2}.$

3) $\langle \Omega_{s,w}, F^* \rangle = * \, D(F, s, k-w-s/2)$ all $F \in S_{2,k}$.

Cor: For $F \in S_{2,k}$, $k \geq 12$,

a) $s(s-1/2)(1-s) * D(F,s,w)$ has holomorphic cont. to $\mathbb{C}^2$.

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b) $\Gamma_2(-s, w + s/2) D(F,s,w)$ is invariant under

$$(s,w) \mapsto (s, k-w)$$

c) $\pi^{-s}\Gamma(s) D(F,s,w)$ is invariant under

$$(s,w) \mapsto (1-s,w)$$

[(a) & (b) already know by work of Maass]

Applications:

- $h \in S^+_{1,k-1/2}(\Gamma_0(4)) \rightsquigarrow F_h \in S_{2,k}$ Sk-lift

$$D(F_h, s, w) = * \underbrace{R(h, \xi, \dots)}_{\text{Rankin-Selberg.}}$$

- $F(Z) = \sum_{T \in J} c(T) e(TZ) \in S_{2,k}$

$$= \sum_{\substack{n>0,\, m>0 \\ D=4\det T>0}} c(n,r,m) e(n\tau_1 + rz + m\tau_2)$$

$$= \sum_{m \geq 1} f_m(\tau_1, z) e(m\tau_2)$$

Each $f_m : \mathfrak{h}_1 \times \mathbb{C} \to \mathbb{C}$ is a Jacobi cusp form of level $\Gamma_1 \ltimes \mathbb{Z}^2$.

Now

$$f_m(\tau,z) = \sum_{r=0}^{2m-1} f_{m,r}(\tau_1)\, \Theta_{m,r}(\tau_1, z)$$

$$f_{m,r}(\tau_1) = \sum_{D=1}^{\infty} c\left(\frac{D+r^2}{4m}, r, m\right) e\left(\frac{D}{4m}\tau_1\right)$$

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$$f_m \rightsquigarrow \Big(\ldots, \underbrace{* \sum_{D=1}^{\infty} c\Big(\frac{D+r^2}{4m}, r, m\Big) D^{-s}}_{\Lambda_r(f_m, s)}, \ldots \Big)_{r=0}^{2m-1}$$

Prop: $$D(F, s, w - s/2) = \sum_{m=1}^{\infty} \frac{\sum_{r=0}^{2m-1} \Lambda_r(f_m, w-s)}{m^{w}}.$$

$$= \sum_{m=1}^{\infty} \frac{\Lambda_0(f_m, k - w + s - 1/2)}{m^{w}}$$

Note need all of $c(n,r,m)$ to define $D(F,s,w-s/2)$, but this result shows one only really needs the $c(n,0,m)$ to define $D(F,s,w-s/2)$.

Thm: Let $F, \tilde{F} \in S_{2,k}$. If $c(T) = \tilde{c}(T)$ for all but finitely many $T = \begin{pmatrix} n & 0 \\ 0 & p \end{pmatrix}$ s.t. p is an odd prime and n is odd square free pos. integer, Then $F = \tilde{F}$.

Previously: Similar results w/. $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ s.t. $(n,r,m) = 1$ by Zagier '81.

- $T = \begin{pmatrix} * & * \\ * & p \end{pmatrix}$ where p odd prime, $-4 \det T$ is odd fund. disc. (Saha '13)