Special Values of Gamma and Zeta Functions:

$$\begin{tikzcd}[column sep=small]
 & \mathbb{C}_\infty = \widehat{\overline{k_\infty}} \arrow[d, dash] & \\
 & \overline{k_\infty} \arrow[dl, dash] \arrow[dr, dash] & \\
k_\infty = \mathbb{F}_q((\tfrac{1}{\theta})) \arrow[dr, dash] & & \bar{k} \arrow[dl, dash] \\
 & k &
\end{tikzcd}$$

$$A = \mathbb{F}_q[\theta] \subseteq k = \mathbb{F}_q(\theta)$$

Carlitz Module:

$$C : \mathbb{F}_q[\theta] \longrightarrow k[\tau]$$

$$\theta \longmapsto C_\theta = \theta + \tau$$

$x \in \mathbb{C}_\infty$

$$\hookrightarrow C_\theta(x) = \theta x + x^q$$

Riemann Zeta Function:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \quad s > 1$$

If $k \in 2\mathbb{Z}$, $k > 1$, then

$$\zeta(k) = r_k \pi^k, \quad r_k \in \mathbb{Q}^\times$$

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}$$

What about $\zeta(k)$, $k \in 2\mathbb{Z} + 1$? Much less is known here.

Note: k even $\Rightarrow \zeta(k) \notin \overline{\mathbb{Q}}$, but if k odd we don't really know

We know: $\zeta(3) \notin \mathbb{Q}$ (Apéry 1970's)

Infinitely many $\zeta(k)$, k odd, are irrational. (Rivoal 2000)

Among $\zeta(5), \zeta(7), \zeta(11), \dots, \zeta(21)$ at least one is irrational. (Rivoal - Zudilin)

Expectation / Conjecture: The numbers

$$\{\pi\} \cup \{\zeta(3), \zeta(5), \zeta(7), \dots\}$$

are algebraically independent over $\overline{\mathbb{Q}}$.

The Carlitz Zeta Function:

$$\zeta_C(k) = \sum_{\substack{a \in A \\ a \text{ monic}}} \frac{1}{a^k} \in k_\infty \qquad (k \in \mathbb{N})$$

$$= \prod_{\substack{f \in A \\ f \text{ monic, irred}}} \left(1 - \frac{1}{f^k}\right)^{-1}.$$

$$\zeta_C(1) = \sum_{a \in A_+} \frac{1}{a} = \sum_{i=0}^{\infty} \left(\sum_{\substack{a \in A_+ \\ \deg(a) = i}} \frac{1}{a} \right)$$

A_+ means monic

$$= 1 + \underbrace{\overbrace{\left(\frac{1}{\theta} + \frac{1}{\theta + 1} + \dots + \frac{1}{\theta + \alpha} \right)}^{\text{deg 1 sum}}}_{\alpha \in \mathbb{F}_q} + \dots$$

$$= \frac{-1}{\prod_{\alpha \in \mathbb{F}_q} (\theta + \alpha)} = \frac{-1}{\theta^q - \theta}$$

Though more complicated, one can work out similar expressions for the other terms.

Carlitz Exponential:

$$\exp_C(z) = \sum_{i\geq 0} \frac{z^{q^i}}{D_i}, \quad D_i \in k, \quad D_0 = 1$$

$$\exp_C(\theta z) = C_\theta(\exp_C(z))$$
$$= \theta \exp_C(z) + \exp_C(z)^q$$

$$\Rightarrow \frac{\theta^{q^i}}{D_i} = \frac{\theta}{D_i} + \frac{1}{D_{i-1}^q}$$

$$\Rightarrow \quad D_i = (\theta^q - \theta)\, D_{i-1}^q$$

If we let $[i] := \theta^{q^i} - \theta$, we have

$$D_i = [i][i-1]^q \cdots [1]^{q^{i-1}}$$
$$= (\theta^{q^i} - \theta) \cdots (\theta^{q^i} - \theta^{q^{i-1}})$$

= product of all monic polys. in $\mathbb{F}_q[\theta]$ of degree i.

The degree 2 sum in $\zeta_C(1)$ is

$$\frac{1}{(\theta^{q^2}-\theta)(\theta^q-\theta)} = \frac{1}{[1][2]}.$$

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Carlitz Logarithm:

$$\log_C(z) = \sum_{i \geq 0} (-1)^i \frac{z^{q^i}}{L_i}$$

This is defined so that $\log_C \circ \exp_C = \exp_C \circ \log_C = z$.

We also have

$$\theta \log_C(z) = \log_C(\theta z + z^q)$$

$$\frac{\theta}{L_i} = \frac{\theta^{q^i}}{L_i} - \frac{1}{L_{i-1}}$$

$$L_i = (\theta^{q^i} - \theta) L_{i-1}$$

$\Rightarrow$

$L_i = [1][2]\cdots[i] =$ least common multiple of all monic polynomials of degree i

One can show:

$$\zeta_C(1) = \sum_{i=0}^{\infty} \frac{(-1)^i}{L_i} = \log_C(1).$$

In fact, for $n < q-1$, $n \in \mathbb{N}$,

$$\zeta_C(n) = \sum_{i=0}^{\infty} \frac{(-1)^{in}}{L_i^n} = \log_C^{[n]}(1)$$

where

$$\log_C^{[n]}(z) = \sum_{i \geq 0} \frac{(-1)^{in}}{L_i^n} z^{q^i}$$

Arithmetic Factorial:

Recall for $n \in \mathbb{N}$, $n! = \prod_p p^{n_p}$ where $n_p = \sum_{i \geq 1} \lfloor \frac{n}{p^i} \rfloor$.

$p \in \mathbb{Z}$ primes.

Consider for $n \in \mathbb{N}$

$$\Pi(n) = \prod_{\substack{f \in A_+ \\ f \text{ irred.}}} f^{n_f}.$$

where

$$n_f = \sum_{i \geq 1} \left\lfloor \frac{n}{N(f)^i} \right\rfloor, \quad N(f) = q^{\deg f}.$$

$\Rightarrow$

$$\Pi(n) = \prod_{i \geq 0} D_i^{n_i} \quad \text{where} \quad n = \sum n_i q^i.$$

So we have a function $\Pi : \mathbb{N} \longrightarrow \mathbb{F}_q[\theta]$. We can extend this function to

$$\overline{\Pi}(n) = \prod_{i \geq 0} \overline{D}_i^{\,n_i}$$

for $n \in \mathbb{Z}_p$, $n = \sum n_i q^i$, $\overline{\Pi} : \mathbb{Z}_p \longrightarrow k_\infty$, setting

$$\overline{D}_i = D_i / \theta^{\deg D_i} \leftarrow \text{1-unit in } k_\infty$$

Normalizing like this ensures the product will converge.

<u>Arithmetic Gamma Function:</u>

$$\Gamma(n) = \overline{\Pi}(n-1)$$

<u>Thakur:</u> $\Gamma(n)\Gamma(1-n) = \Gamma(0)$ 1-unit part of $\tilde{\pi}$.

$$= \tilde{\overline{\pi}}$$

$$\Gamma\left(1 - \frac{a}{q-1}\right) = \left(\overline{\tilde{\pi}}\right)^{a/q-1} \qquad 0 \leq a < q-1.$$

So if $2 \nmid q$, $\Gamma(\frac{1}{2}) = \sqrt{\tilde{\pi}}$.

<u>Note:</u> $\exp_C(z) = \sum_{i \geq 0} \frac{z^{q^i}}{\pi(q^i)}$.

Carlitz-Bernoulli numbers.

Write $\frac{z}{\exp_C(z)} = \sum_{i=0}^{\infty} \frac{B_i}{\pi(i)} z^i$, $B_i \in \mathbb{F}_q(\theta)$.

$$\frac{z}{\exp_C(z)} = \frac{z\,(\exp_C(z))'}{\exp_C(z)} = 1 - \sum_{\substack{a \in A \\ a \neq 0}} \frac{z/\tilde{\pi}a}{1 - z/\tilde{\pi}a}$$

(since $\ker(\exp_C(z)) = \tilde{\pi}\,\mathbb{F}_q[\theta] \Rightarrow \exp_C(z) = z \prod_{\substack{a \in A \\ a \neq 0}} (1 - \frac{z}{\tilde{\pi}a})$)

geo series

$$= 1 - \sum_{n=1}^{\infty} \underbrace{\sum_{\substack{a \in A \\ a \neq 0}} \left(\frac{z}{\tilde{\pi}a}\right)^n}_{\text{vanishes if } (q-1) \nmid n}$$

$$= 1 - \sum_{\substack{n \geq 1 \\ (q-1) \mid n}} \left(\sum_{\substack{a \in A \\ a \neq 0}} \frac{1}{a^n}\right)\left(\frac{z}{\tilde{\pi}}\right)^n$$

$(q-1)(-1) = 1$

$$= 1 + \sum_{\substack{n \geq 1 \\ (q-1) \mid n}} \left(\sum_{a \in A_+} \frac{1}{a^n}\right)\left(\frac{z}{\tilde{\pi}}\right)^n$$

$$= 1 + \sum_{\substack{n \geq 1 \\ (q-1) \mid n}} \zeta_C(n) \left(\frac{z}{\tilde{\pi}}\right)^n.$$

If $(q-1) \mid n$, then $\zeta_C(n) = \frac{B_n}{\pi(n)} \tilde{\pi}^n$. Because of this sometimes one says "n is even" if $(q-1) \mid n$.

<u>Thakur:</u> $A = \mathbb{F}_3[\theta, \eta] / (\eta^2 - \theta^3 + \theta + 1)$ (class number 1)

$$\zeta_A(1) = \sum_{\substack{a\in A \\ a \text{ monic}}} \frac{1}{a} = \log_\rho(\eta - 1)$$

rank 1 Drinfeld A-module

Ex: $A = \mathbb{F}_3[\theta,\eta]/(\eta^3 - \theta^3 + \theta^2 + \theta)$ (class number 2)

$$\zeta_A(1) = \log_{\rho'}(1 - \sqrt{\theta}(1+\mu)) - \frac{1}{\sqrt{\theta}} \log_{\rho'}(\sqrt{\theta} + \theta - \mu^3 - \eta^3 - \sqrt{\theta}/\mu - (\sqrt{\theta})^9 - (\sqrt{\theta})^3 + 1)$$

where $\mu = 1 + \theta + \eta/\sqrt{\theta}$ is a fundamental unit in the Hilbert class field.

In general, $\zeta_C(n) = k$-linear combination of $\log_C^{[n]}(\alpha)$'s $\alpha \in k$. This is due to Anderson-Thakur. (This is for all $n \in \mathbb{N}$.) This can be done explicitly.

Thm (Chang, P., Thakur, Yu 2008): The transcendence degree over $\bar{k}$ of $\{\zeta_C(1), \ldots, \zeta_C(n)\} \cup \{\Gamma(\frac{1}{q^\ell - 1}), \ldots, \Gamma(\frac{q^\ell - 2}{q^\ell - 1})\} \cup \{\tilde{\pi}\}$ is

$$n - \lfloor \tfrac{n}{p} \rfloor - \lfloor \tfrac{n}{q-1} \rfloor + \lfloor \tfrac{n}{p(q-1)} \rfloor + \ell.$$