MATH 108H — SECOND MIDTERM EXAM

April 7, 2010

NAME: ____

1. Do not open this exam until you are told to begin.
2. This exam has 11 pages including this cover. There are 6 problems.
3. Write your name on the top of EVERY sheet of the exam!
4. Do not separate the pages of the exam.
5. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so I will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work for each exercise so that I can see not only the answer but also how you obtained it.
7. You may use your calculator. However, you are NOT allowed to use it to evaluate integrals or take derivatives. If you use it in a significant way, explain how you used it. (For example, if you use it to graph arc tangent or something comparable to that.)
8. Turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	15	
2	25	
3	10	
4	15	
5	15	
6	20	
TOTAL	100	

Name: ____

1. (3 points each) Circle the correct answer:

(a) If $y = f(x)$ is a periodic function, generally the best way to get an overall approximation to the function is to use a *Taylor/Fourier* polynomial.

(b) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ must *converge/diverge/don't know.*

(c) *True/False*: If $p(x)$ is a probability distribution, then $\int_{-\infty}^{\infty} p(x)dx = 1$.

(d) *True/False*: If $y = f(x)$ is a continuous function and $f(n) = a_n$ for all n, then if $\int_c^{\infty} f(x)dx$ converges so does $\sum_{n=1}^{\infty} a_n$.

(e) If $P_2(x) = 1 + 3(x-2) - 4(x-2)^2$ is the second degree Taylor polynomial of $y = f(x)$ around $x = 2$, then the function $f(x)$ is necessarily *concave up/concave down/don't know* at $x = 2$.

2. (5 points each) Determine if the following converge or diverge. Be sure to justify your answer to receive any credit!

(a) The sequence given by $a_n = 1/n$.

Name: ____

(b) The series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n}.$$

(c) The series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 1}}.$$

Name: ____

(d) The series

$$\sum_{n=0}^{\infty} \frac{5^n}{n!}.$$

(e) The series

$$\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n.$$

Name: ____

3. (5 points each)As you ride your bike you drop your sunglasses. You are not sure where you dropped them. Suppose the probability density function $p(x)$ for having dropped the sunglasses x kilometers from home is given by

$$p(x) = 2e^{-2x} \quad \text{for } x \geq 0.$$

(a) What is the probability you dropped the sunglasses within 1 kilometer of home?

(b) At what distance y from home is the probability that you dropped the sunglasses within y kilometers of home equal to 0.99?

Name: ____

4. (5 points each) **(a)** Calculate the Taylor series of $\ln(1+x)$ around $x = 0$ and determine where it converges. (Hint: $\int \frac{1}{1+x} dx = \ln|1+x| + C$.)

Name: ____

(b) Use a degree 3 Taylor polynomial to approximate $\ln(0.9)$.

(c) Bound the error in the approximation you found in part (b).

Name: ____

5. (5 points each) Recall that if you deposit A dollars in a bank account earning interest at a rate of $r\%$ per year, compounded annually, then n years from now the account will contain B dollars where B is given by

$$B = A(1 + r)^n.$$

The *present value* of a deposit made n years in the future is the amount of money one would need to deposit today in order to have that amount of money in n years. From the formula above, we see the present value of B dollars is given by $A = B(1 + r)^{-n}$.

(a) Suppose, starting today, you deposit \$100 in the bank each year. What is the present value of each of the first 3 deposits if the interest rate is 4%? What is the total present value of these deposits?

Name: ____

(b) What is the total present value of the first 50 deposits?

(c) Such a deposit pattern is called a perpetuity if it continues indefinitely. If these deposits are a perpetuity, what is the total present value of this perpetuity?

Name: ____

6. (10 points each) According to Einstein's theory of relativity, the kinetic energy of a body traveling at a velocity of v that has mass m_0 at rest is given by

$$K = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2$$

where c is the speed of light.

(a) Show by using a Taylor series how one can recover the classical formula that says if v is much smaller than c then

$$K \approx \frac{1}{2} m_0 v^2.$$

Hint: $x = \frac{v^2}{c^2} < 1$.

Name: ____

(b) If you include one more term in the approximation and $v = 0.0002c$, by what percentage does the approximation in part (a) differ from this approximation?