MATH 206H — FINAL EXAM

December 6, 2010

NAME: ______________________________

1. Do not open this exam until you are told to begin.
2. This exam has 14 pages including this cover. There are 6 problems.
3. Write your name on the top of EVERY sheet of the exam!
4. Do not separate the pages of the exam.
5. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so I will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work for each exercise so that I can see not only the answer but also how you obtained it.
7. You may only use a basic calculator. You do not need to simplify complicated expressions you would normally type into your calculator.
8. Turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	20	
2	10	
3	10	
4	10	
5	30	
6	20	
TOTAL	100	

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1. (10 points each) **(a)** Find the tangent plane to the surface $x^3z + xy + z^2y = 7$ at the point $(1, 1, 2)$.

(b) Give the line of intersection of the plane you found in part (a) and the plane $-x+y-2z = -4$.

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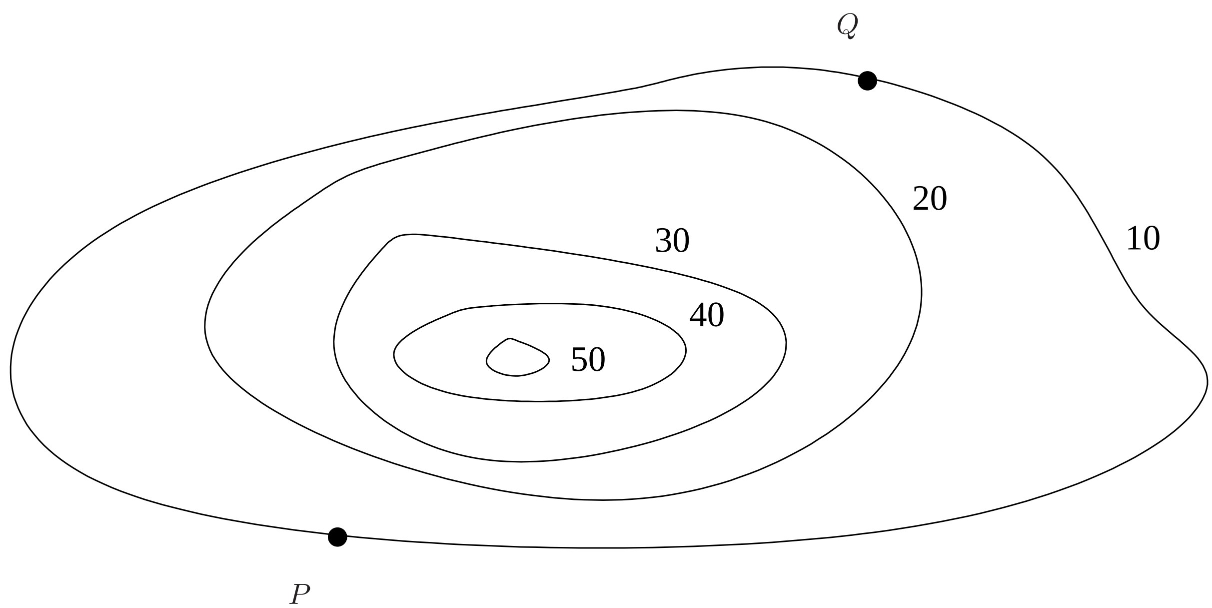
2. (10 points) On the contour plot below, Karen is standing at point P and Professor Hinkle is standing at point Q when they spot the magic hat at the top of the hill. They each decide to take the path of steepest ascent to the top of the hill. Draw each of their paths to the top. Give an explanation of how you arrived at the paths drawn.



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3. (10 points) Karen builds Frosty out of the snow that consists of magical Christmas snow, regular snow, and the dreaded "yellow snow." Frosty's resistance to melting is given by

$$R(c, s, y) = 2cs + 3cy + 2sy$$

where c, s, and y are the proportions of Christmas snow, regular snow, and yellow snow respectively. Use the fact that $c + s + y = 1$ to calculate the proportions Karen should use in order to make Frosty as heat resistant as possible.

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4. (10 points) After leaving the train to the north pole Frosty and Karen wander looking for a place to warm Karen back up. The temperature at a point (x, y) is given by $T(x, y)$, measured in degrees Celsius. Their path is given by $x = \sqrt{1+t}$, $y = 2 + \frac{t}{7}$, where x and y are measured in meters and t is in seconds since leaving the train. The temperature function satisfies $\frac{\partial T}{\partial x}(6, 7) = 0.04$ and $\frac{\partial T}{\partial y}(6, 7) = -0.14$. How fast is the temperature changing on their path after 35 seconds? Is this a better path for Frosty or Karen? Be sure to justify your answer.

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5. (5 points each) Frosty and Karen find a greenhouse to enter to help warm Karen. However, Professor Hinkle locks them in the greenhouse and Frosty proceeds to melt, forming a puddle of water on the floor. The velocity vector field near a drain in the greenhouse floor is given, for x, y, z in cm, by

$$\vec{F}(x,y,z) = -\frac{y+xz}{(z^2+1)^2}\hat{\imath} - \frac{yz-x}{(z^2+1)^2}\hat{\jmath} - \frac{1}{z^2+1}\hat{k} \qquad \text{cm/sec.}$$

(a) The drain in the bathtub is a disk in the xy-plane with center at the origin and radius 1 cm. Find the rate at which melted Frosty is leaving the greenhouse (that is, the rate at which water is flowing through the disk.) Give units for your answer.

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(b) Find the divergence of $\vec{F}$.

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(c) Find the flux of the water through the hemisphere of radius 1, centered at the origin, lying below the xy-plane and oriented downward.

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(d) Find $\oint_C \vec{G} \cdot d\vec{r}$ where C is the edge of the drain, oriented clockwise when viewed from above, and where

$$\vec{G}(x, y, z) = \frac{1}{2}\left(\frac{y}{z^2+1}\hat{i} - \frac{x}{z^2+1}\hat{j} - \frac{x^2+y^2}{(z^2+1)^2}\hat{k}\right).$$

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(e) Calculate $\text{curl}\,\vec{G}$.

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(f) Explain why your answers to parts (c) and (d) are equal.

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6. (5 points each) Santa informs Karen that since Frosty is at least partially made of Christmas snow, he can bring him back to life. However, Santa has an evil gleam in his eye. He says that in order to help Frosty, Karen must first answer the following application problems. He allows Karen one lifeline, which she uses to call you. Frosty's life depends on your abilities here. Please don't let Frosty die!
Recall that the electric field from a point charge q at a distance of r is given by

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\vec{r}}{r^3}$$

where we write $r = |\vec{r}|$. In this problem we consider a wire (a line for our purposes) of length $2L$ with charge density σ C/m. (We do not assume σ is constant for part (a)!)

(a) Give an integral that gives the electric field produced by the charged wire at a distance of a above the center of the wire.

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(b) For the rest of the problem assume σ is a constant. Evaluate the integral you found in part (a). A substitution of the form $x = r\tan(\theta)$ may come in handy at some point in the problem.

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(c) Find the electric field of an infinitely long wire by taking the limit as $L \to \infty$ of your answer in (b).

(d) Recall Gauss' Theorem gave

$$\iint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

where Q_{encl} is the total charge enclosed by the closed surface S. Use this to calculate the electric field of an infinitely long line and note that you recover the answer you obtained in part (c).