MATH 2060 — FINAL EXAM

April 29, 2014

NAME: ____

1. Do not open this exam until you are told to begin.
2. This exam has 15 pages including this cover. There are 5 problems.
3. Write your name on the top of EVERY sheet of the exam at the START of the exam!
4. Do not separate the pages of the exam.
5. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so I will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work for each exercise so that I can see not only the answer but also how you obtained it.
7. You may use a non-graphing calculator. You are NOT allowed to use it to do anything significant such as integrating, taking derivatives, etc.
8. Turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	15	
2	10	
3	10	
4	25	
5	40	
TOTAL	100	

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1 (5 points each) Let $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = -\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

(a) What is the angle between $\mathbf{v}$ and $\mathbf{w}$?

(b) Calculate $\mathbf{v} \times \mathbf{w}$.

(c) Give an equation of a line through the point $(2, 7, 9)$ in the direction of $\mathbf{v}$.

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2. (5 points each) The van der Waals equation for n moles of a gas is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas.

(a) Calculate $\frac{\partial T}{\partial P}$ and $\frac{\partial T}{\partial V}$.

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(b) Suppose that the pressure and volume are changing with time. At time t_0 we have $\frac{dP}{dt}|_{t_0} = c$ and $\frac{dV}{dt}|_{t_0} = d$. What is $\frac{dT}{dt}|_{t_0}$?

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3. (10 points) Find the equation of the tangent plane at the point $(2, -1, 3)$ to the ellipsoid $\frac{x^2}{4}+y^2+\frac{z^2}{9} = 3$. Sketch the ellipsoid and the tangent vector.

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4. (5 points each) Let C be the curve given by the triangle with vertices $(0,1,0)$, $(1,2,5)$, and $(-1,2,3)$ oriented counterclockwise. Let $\mathbf{F}(x,y,z) = 2y\mathbf{i} + (x^2 - z^2)\mathbf{j} + 3xy\mathbf{k}$.

(a) Give a parameterization of C. (It is fine to break C into three line segments and parameterize each of the pieces instead.) Evaluate $\mathbf{F}(\mathbf{r}(t))$ on each of the segments.

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(b) Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly.

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(c) Find a normal vector to the plane the triangle lies in.

(d) Calculate $\nabla \times \mathbf{F}$.

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(e) Give a surface integral that is equal to $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and evaluate this surface integral (don't just say it is equal to what you've already calculated!)

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5. (20 +5 + 10 + 5 points) Suppose we have a charged particle with charge q located at Q. Let P be a point so that ρ is the vector from Q to P. Then the electric field at P due to Q is given by $\mathbf{E}(\rho) = \frac{q}{4\pi\varepsilon_0}\frac{\rho}{|\rho|^3}$ where ε_0 is the electric constant. (Note here ρ is a vector; I can't get it to bold correctly.)

(a) Consider a point P located at $(0, 0, z_0)$ with $z_0 \neq 0$ and a charged wire lying along the x-axis from $(-a, 0, 0)$ to $(a, 0, 0)$ with constant charge density q Coloumbs per meter. Show the electric field at P due to the charged wire is

$$\mathbf{E}_{\text{wire}}(0, 0, z_0) = \frac{2qa}{4\pi\varepsilon_0 z_0(a^2 + z_0^2)^{1/2}}\mathbf{k}.$$

(Hint: Slice the charged wire into small pieces Δx and consider the charge due to the slice at x and $-x$ at the same time. This should give there is only nonzero electric field in the $\mathbf{k}$ component.)

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(b) Now suppose the wire is infinite in length. (One often makes this simplification if the value z_0 is small compared to x_0 since the electric field coming from far away is negligible due to the inverse square law.) Show the electric field at P due to the infinite wire is

$$\mathbf{E}_{\text{wire}}(0,0,z_0) = \frac{2q}{4\pi\varepsilon_0 z_0}\mathbf{k}.$$

(You may use the answer from part (a)!)

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(c) Note that in part (b), since the wire is now assumed to be infinite, this calculation applies to any point $P = (x, y, z)$ since the symmetry used in part (a) is true now for EVERY point. This means that the electric field at a point $P = (x, y, z)$ that does not lie on the wire is

$$\mathbf{E}_{\text{wire}}(x, y, z) = \frac{2q}{4\pi\varepsilon_0 (y^2 + z^2)^{3/2}} (y\mathbf{j} + z\mathbf{k}).$$

Now given a finite length wire, as long as the point P is close to the wire with respect to the length of the wire, we can use this equation as a good approximation to the electric field at P due to the finite length wire. We will now derive Gauss' law for a charged wire. Let S be the cylinder $y^2 + z^2 = R^2$ for $-b \leq x \leq b$ along with the two ends of the cylinder so that we have a closed surface that completely contains the wire. Show that

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \frac{2qb}{\varepsilon_0}.$$

Note here that $2qb$ is the total charge enclosed by S. (Hint: Use the symmetry of the vector field with relation to the normal vector of the surfaces.)

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(d) Let P be any point not on the wire. We know that the divergence of the electric field at P due to any single point charge on the wire is 0 from class. We can therefore say the divergence at P due to the entire charged wire is 0. Use this to prove Gauss' law, namely, that if S is ANY surface enclosing the wire, then

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \frac{2qb}{\varepsilon_0}.$$