# MATH 142 — MIDTERM 2

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NAME: Solutions

1. (5+10 points) Let $f(x) = 4 + \sqrt{3 + 2x}$.

   (a) What is the domain of $f(x)$?

   In order for $x$ to be in the domain of $f(x)$ we must have $3 + 2x \geq 0$. In other words, the domain consists of $x \geq -3/2$.

   (b) Find $f^{-1}(x)$ (including its domain!)

   Set $y = 4 + \sqrt{3 + 2x}$. We now solve this for $x$:

$$\begin{aligned} y &= 4 + \sqrt{3 + 2x} \\ y - 4 &= \sqrt{3 + 2x} \\ (y - 4)^2 &= 3 + 2x \\ x &= \frac{1}{2}((y - 4)^2 - 3). \end{aligned}$$

   We now interchange $x$ and $y$ and have

$$f^{-1}(x) = \frac{1}{2}((x - 4)^2 - 3).$$

   The domain for this function is all real numbers.

2. (10 points each) Solve the following equations for $x$:

   (a) $e^{2x-2} = 4e^x$

   We apply the natural logarithm to both sides to obtain

$$\ln(e^{2x-2}) = \ln(4e^x).$$

   We have $\ln(e^{2x-2}) = 2x - 2$ and $\ln(4e^x) = \ln(4) + \ln(e^x) = \ln(4) + x$. Thus,

$$\begin{aligned} 2x - 2 &= \ln(4) + x \\ x &= \ln(4) + 2. \end{aligned}$$

(b) $\ln(x) + \ln(x-1) = 1$

We have $\ln(x) + \ln(x-1) = \ln(x(x-1)) = \ln(x^2 - x)$. Thus, we want to solve the equation $\ln(x^2 - x) = 1$. We exponentiate each side to obtain $e^{\ln(x^2-x)} = e^1$. Thus, $x^2 - x = e$. This is a quadratic equation $x^2 - x - e = 0$, so we use the quadratic formula to obtain

$$x = \frac{1 \pm \sqrt{1+4e}}{2}.$$

Note that $\frac{1-\sqrt{1+4e}}{2} < 0$ and since $\ln(x)$ has domain $x > 0$, this cannot be a solution. Thus, the only solution is $x = \frac{1+\sqrt{1+4e}}{2}$.

3. (5 points each) Evaluate the following integrals:

(a) $\int xe^{x^2} dx$

We set $u = x^2$ so $du = 2xdx$. This gives

$$\begin{aligned}\int xe^{x^2} dx &= \frac{1}{2}\int e^u du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{x^2} + C.\end{aligned}$$

(b) $\int_e^{e^2} \frac{\ln(x)}{x} dx$

We set $u = \ln(x)$ so $du = \frac{1}{x}dx$. The limits change to start at $\ln(e) = 1$ to $\ln(e^2) = 2$. This gives

$$\begin{aligned}\int_e^{e^2} \frac{\ln(x)}{x} dx &= \int_1^2 u du \\ &= \frac{1}{2}u^2 \mid_1^2 \\ &= \frac{1}{2}(4-1) \\ &= \frac{3}{2}.\end{aligned}$$

(c) $\displaystyle\int_1^2 \frac{3^x}{3^x+2}dx$

We set $u = 3^x + 2$ so $du = 3^x \ln(3)dx$. The limits change to being from $3^1 + 2 = 5$ to $3^2 + 2 = 11$. This gives

$$\begin{aligned}\int_1^2 \frac{3^x}{3^x+2}dx &= \frac{1}{\ln(3)} \int_5^{11} \frac{1}{u}du \\ &= \frac{1}{\ln(3)} \ln|u| \,|_5^{11} \\ &= \frac{\ln(11) - \ln(5)}{\ln(3)} \\ &= \frac{\ln(11/5)}{\ln(3)}.\end{aligned}$$

(d) $\displaystyle\int (2^x + x^2)dx$

We have

$$\begin{aligned}\int (2^x + x^2)dx &= \int 2^x dx + \int x^2 dx \\ &= \frac{2^x}{\ln(2)} + \frac{x^3}{3} + C.\end{aligned}$$

(e) If $f(0) = 3$, $f(\pi/2) = 4$, and $f(1) = 10$, evaluate $\displaystyle\int_0^{\pi/2} f'(\cos(x))\sin(x)dx$.

Let $u = \cos(x)$ so $du = -\sin(x)$ and the limits change from 0 to $\pi/2$ to being from $\cos(0) = 1$ to $\cos(\pi/2) = 0$. Thus, we have

$$\begin{aligned}\int_0^{\pi/2} f'(\cos(x))\sin(x)dx &= -\int_1^0 f'(u)du \\ &= f(1) - f(0) \\ &= 10 - 3 \\ &= 7.\end{aligned}$$

Note we used the fundamental theorem of calculus above.

4. (5 points each) Differentiate the following functions:

(a) $f(x) = x\ln(x) - \cos(x^2)$

We use the product rule on $x\ln(x)$ and the chain rule on $\cos(x^2)$:

$$\begin{aligned} f'(x) &= 1\ln(x) + \frac{x}{x} + \sin(x^2)2x \\ &= 1 + \ln(x) + 2x\sin(x^2). \end{aligned}$$

(b) $f(t) = 5^{\sqrt{t}}$

Set $y = f(t) = 5^{\sqrt{t}}$. Taking the natural logarithm of each side gives $\ln(y) = \ln(5^{\sqrt{t}}) = \sqrt{t}\ln(5)$. It is important to note that $\ln(5)$ is a constant! Using implicit differentiation to differentiate both sides we obtain $\frac{1}{y}\frac{dy}{dx} = \ln(5)\frac{1}{2\sqrt{t}}$. Thus,

$$\frac{dy}{dx} = y\frac{\ln(5)}{2\sqrt{t}} = 5^{\sqrt{t}}\frac{\ln(5)}{2\sqrt{t}}.$$

(c) $f(z) = (\sin z)^{\ln(z^2)}$

Set $y = (\sin z)^{\ln(z^2)}$. We take natural logarithms of each side to obtain $\ln(y) = \ln((\sin z)^{\ln(z^2)}) = \ln(z^2)\ln(\sin(z)) = 2\ln(z)\ln(\sin(z))$. We now differentiate each side to obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{z}\ln(\sin(z)) + 2\ln(z)\frac{\cos(z)}{\sin(z)}.$$

Thus, we have

$$\begin{aligned} \frac{dy}{dx} &= y\left(\frac{2}{z}\ln(\sin(z)) + 2\ln(z)\frac{\cos(z)}{\sin(z)}\right) \\ &= (\sin z)^{\ln(z^2)}\left(\frac{2}{z}\ln(\sin(z)) + 2\ln(z)\frac{\cos(z)}{\sin(z)}\right). \end{aligned}$$

(d) $g(x) = \int_3^{e^{x^2}} \cos(t)dt$

One can do this two ways: one can evaluate the integral and then take the derivative or one can use the fundamental theorem of calculus. Using the fundamental theorem of calculus and the chain rule we obtain

$$g'(x) = \cos(e^{x^2})e^{x^2}2x.$$

(e) $h(v) = \tan(4^{v^2})$

This requires chain rule:

$$h'(v) = \sec^2(4^{v^2}) \ln(4) 4^{v^2} 2v.$$

5. (7+8 points) Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{a}{1 + be^{-kt}}$$

where $a$, $b$, and $k$ are positive constants.

(a) Find $\lim_{t\to\infty} p(t)$.

Since $k$ is a positive constant, we know that $\lim_{t\to\infty} e^{kt} = \infty$. Thus, $\lim_{t\to\infty} be^{-kt} = \lim_{t\to\infty} \frac{b}{e^{kt}} = 0$. Thus,

$$\lim_{t\to\infty} p(t) = a.$$

(b) Find the rate of spread of the rumor.

The rate is given by $p'(t)$. We have $p(t) = a(1 + be^{-kt})$ so

$$\begin{aligned} p'(t) &= a(-1)(1 + be^{-kt})^{-2} be^{-kt}(-k) \\ &= \frac{abke^{-kt}}{1 + be^{-kt}} \\ &= \frac{abk}{e^{kt} + b}. \end{aligned}$$