Math 333 Problem Set 10
Due: 05/11/16

Be sure to list EVERYONE in the that you talk to about the homework!

1. (a) Prove the set T of matrices of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ with $a, b \in \mathbb{R}$ is a subring of $\text{Mat}_2(\mathbb{R})$.

   (b) Prove the set I of matrices of the form $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ with $b \in \mathbb{R}$ is an ideal in the ring T.

   (c) Show that every coset in T/I can be written in the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + I$.

2. Let R be a ring. Show that the map $\varphi : R[x] \to R$ that sends each polynomial to its constant term is a surjective ring homomorphism.

3. Let F be a field, R a nonzero ring, and $\varphi : F \to R$ a surjective ring homomorphism. Prove that φ is an isomorphism.

4. (a) Let $\varphi : R \to S$ be a surjective homomorphism of rings. Let I be an ideal in R. Prove that $\varphi(I)$ is an ideal in S.

   (b) Is part (a) true if φ is not surjective? Prove it is true or give a counterexample.

5. Let I be an ideal in a ring R. Prove that every element in R/I has a square root if and only if for every $r \in R$ there exists $a \in R$ so that $r - a^2 \in I$.

6. (a) Let I and K be ideals in a ring R with $K \subset I$. Prove that $I/K = \{a + K : a \in I\}$ is an ideal in the quotient ring R/K.

   (b) Prove that $(R/K)/(I/K) \cong R/I$. (Hint: Define a map $\varphi : R/K \to R/I$ given by $\varphi(r + K) = r + I$. Show this is well-defined, a surjective ring homomorphism, and find its kernel.)